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PRINCIPLES

OF

GEOMETRY

FAMILIARLY ILLUSTRATED.

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PRINCIPLES

OF

GEOMETRY

FAMILIARLY ILLUSTRATED, AND APPLIED TO A

VARIETY OF

USEFUL PURPOSES.

DESIGNED FOR THE INSTRUCTION OF YOUNG PERSONS.

BY THE

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PREFACE.

That "a great book is a great evil," is an aphorism the truth of which few pupils will feel disposed to call in question; and that there is no "royal road to geometry," though an old observation, is nearly as true as when it was first uttered; for though several new plans have been proposed, and well executed, the line of road sketched out and finished nearly two thousand years ago, is generally preferred. To shorten the road, to render it as smooth as the materials will admit, and to open to the view of the young traveller frequent glimpses of the rich domains of *science*, into which he has now entered, are objects which the author has kept steadily in view.

The pupil will also find resting-places at the most commanding points of view, with a few of the simpler kinds of gymnastic apparatus, where he may amuse himself for a little, and ascertain whether he is gaining any geometrical strength as he proceeds. To some inexperienced guides this may appear a waste of that valuable time which might be more profitably employed in advancing with apparently more rapid steps towards the Temple of Science. But to those who have frequently conducted youth along this path, those resting-places for amusement

and exercise will appear in a very different light; for what the youthful traveller may seem to lose in time, he does more than gain in intellectual power, which ought to be the great leading object of every guide who undertakes the delightful, though laborious task of training the youthful mind from the weakness of childhood to the vigour of maturer years.

But to quit metaphorical language, it may, perhaps, be necessary to point out still farther the nature and object of this small volume, and the rank it was intended to hold among other works on the same subject. There appearing to the Author no work on Geometry in which theory and practice went hand in hand, and which held an *intermediate* rank between works purely scientific and those of a popular or merely practical nature, induced him to endeavour to supply this desideratum, the want of which must have been severely felt by those who could devote only a very short period to the acquisition of the more useful parts of this science. Had he adhered to a strictly scientific arrangement, it would have been impossible to have introduced any of those practical *numerical* applications which appear at the very commencement of the work. For example, if it be proved that the numerical expression for the third side of a triangle is determined when the numerical expression for the two sides and the angle contained between those sides are given, when the lengths are taken from a *given* scale of equal parts; it has been assumed that the same numerical expression for the third side will be obtained when the triangle is constructed from a larger or

smaller scale of equal parts. This principle or property, which is not demonstrated in all its generality till the pupil has advanced to nearly the end of the Sixth Book of Euclid's Elements, has been assumed and extensively used in the practical applications for the following reasons:

1st. Young pupils take great delight in constructing neat and accurate geometrical figures by scale and compasses. 2ndly. By doing so from the commencement of their studies, they will acquire a habit of doing whatever they are called upon to do with neatness and accuracy. They will thus be gradually prepared for architectural or mechanical drawing, or for the accurate construction of maps and charts. 3rdly, It will be found that pupils understand geometrical and other properties more easily and clearly when expressed in *numbers* than in *lines* or *general* characters. 4thly. The teacher will thus be enabled to answer a question which is almost universally put by pupils (if they are allowed to say what they think) when taught geometry on purely scientific principles, as laid down in the Elements of Euclid. All that may be very true, but what is the use of it?

Unless some such plan as that employed in this volume be adopted, the teacher must give something like the following answer: You will begin to see the use of all this when you have learned the whole of the Elements. The pupil looks at the large volume; he already knows what it is to have attempted the "Ass's Bridge," and unless he possess first-rate talents or great perseverance, it is almost certain that his

courage from that moment will begin to fail. Of the thousands who spend years in the study of some abstract mathematics, how few will be found on trial who have acquired them in such a manner as to be able to apply them to any useful purpose!

In a conversation which I lately had on this subject with an eminent Professor of the University of Cambridge, he told me that he once put a Hadley's Quadrant into the hands of a young man of high mathematical attainments, who did not even know how to measure an angle by that most useful of all instruments.

We could easily point out the very pages where a purely geometrical critic, who scarcely knows any thing of the "delightful task," will pick, and cull, and find fault, because these pages are not what they never were intended to be; but we shall refrain from doing so lest we should fall into the error of writing a *long* preface to a *small* volume.

Another peculiarity in this volume is, that many of the simpler steps in the demonstrations are left to be supplied by the pupil himself, whose attention will thus be constantly kept up, and his reasoning powers continually exerted. These simple steps generally follow the term "why?" the pupil being expected to give the "wherefore."

These are the principal objects which the author has attempted to execute; how far he has succeeded *time* and *teachers* alone can determine.

CONTENTS.

PART I.

THE SIMPLER PROPERTIES OF LINES, ANGLES, AND SURFACES.

CONTENTS.

PRINCIPLES OF GEOMETRY,

FAMILIARLY ILLUSTRATED.

PART I.

CONTAINING THE SIMPLER PROPERTIES OF LINES, ANGLES, AND SURFACES.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

1. MATHEMATICS is that science which treats of whatever can be numbered or measured.

2. Geometry is that branch of mathematics which treats of the properties of lines, angles, surfaces, and solids.

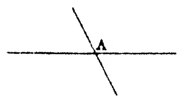
3. If two flat pieces of ground and polished glass be placed above each other, the one surface will coincide with the other in every position. These are called *plane surfaces.* If perfect coincidence be conceived to exist between the planes in every point, we get the idea of a mathematical plane. It is on such planes that geometrical figures are conceived to be constructed.

4. If two fine lines be drawn with a diamond on the surface of each piece of glass, and if the plates be then placed above each other, and the two lines found to coincide in every point, when they will not, in any position of the plates, include a space, each of them is

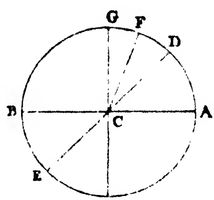
called a *straight* line. If the breadth of the lines be conceived to get less and less till it vanish altogether, we get the idea of a *mathematical line.*

We shall employ the term *line* to denote a straight line.

5. If we consider merely the ends of a line, having no breadth, we get the idea of a point. A mathematical point is that which marks position, but has no magnitude. Hence, if one line be drawn across another, the place A, where they cut each other, is a point.



6. If one of the points of a pair of compasses be fixed, whilst the other is made to turn round, the revolving point will trace out the *circumference* of a *circle*. The fixed point is called the *centre*. Any line, C A, drawn from the centre to the circumference, a *radius;* a line, B A, passing through the centre and having its extremities in the circumference, a *diameter*. Any portion, A D, of the circumference is called an arc, and the straight line joining D A its *chord*.



7. If the circumference be conceived to be divided into 360 equal parts, each part is called a *degree*. Each degree is again supposed to be divided into 60 *minutes*, and each minute may again be conceived to be divided into 60 *seconds*.

Degrees, minutes, and seconds, are thus written: $20° \, 40' \, 25''$.

8. If a line be made to turn round the point C, it will have different positions or inclinations to C A.

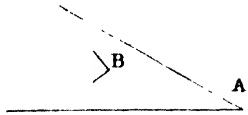
The inclination of one line to another is called an *angle*, and the point from which they diverge, the *angular point*. If the circumference be divided into 360 equal parts, and lines drawn from the centre to each of those divisions, the angle formed by every adjacent pair is called an angle of one degree. It is obvious that all these angles are equal to each other; for if one of them be conceived to be placed on another, the arcs, and consequently the lines forming the angles, will coincide. Hence angles may be compared with each other by comparing the number of degrees contained in the intercepted arcs of a circle described from the angular point as a centre. Thus if the arc A D contain 20°, and F D 5°, the angle A C D will be four times the angle D C F.

An angle is named by means of three letters, one of them being placed at the angular point, and the other two at any points in the two lines, and so arranged that the letter which is at the angular point shall be *between* the other two. Thus the angle formed by the lines, C A, C D, is called the angle A C D or D C A. The following sign $\angle$ is often employed to denote an angle, thus, $\angle$ A C D. When there is only one angle, it may be named by the letter placed at the angular point.

9. When the revolving line comes to the position C G, so that the arc A G is equal to B G, or 90° each, and consequently the angle A C G equal to B C G, each of those angles is called a *right angle*; the line G C is said to be perpendicular to B A.

10. An angle, such as A C D, which is less than a right angle, is called an *acute angle*. An angle greater than a right angle, as B C D, is called an *obtuse angle*.

EXERCISE.—Which of the annexed angles, A or B, is largest?



11. From those definitions, it is obvious that the sum of all the angles which can be formed on one side of a straight line, as A C D, D C F, F C G, G C B, will be equal to two right angles, or 180 degrees. The same may be said of the other side of the line. Hence the sum of all the angles formed by lines diverging from the same point, is equal to 4 right angles, or 360 degrees.

12. From the preceding definitions, it also follows that equal arcs of the same circle must have equal chords. If the arc A D be equal to the arc B E, then the chord of A D will be equal to the chord of B E. For if the arc B E be placed on the arc A D, they will obviously coincide, as will also their chords.

13. Parallel lines are those which lie on the same plane, and which, being produced ever so far both ways, do not meet.

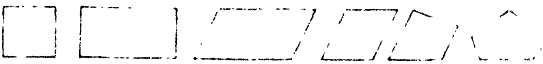
14. A figure, in geometry, is a space of any form inclosed by lines. If it be inclosed by three lines, it is called a *triangle;* if the lines be equal, it is called an *equilateral* triangle; if two of the lines be equal, an *isosceles* triangle; if they be unequal, it is said to be *scalene.* It is *right angled, obtuse angled,* or *acute angled,* according as it has one of its angles right, obtuse, or all of them acute.

15. A figure bounded by *four* straight lines, is called

If the opposite sides be paral-

lel, it is called a *parallelogram*. A right angled parallelogram, is called a *rectangle*. A *square* is a rectangle, having all its sides equal. If the opposite sides of a four-sided figure be equal, it is called a *rhomboid*. If all the sides be equal, a *rhombus*. A figure having more than four sides, is called a *polygon*.

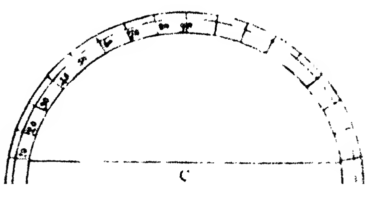
EXERCISE.—Name the following figures.



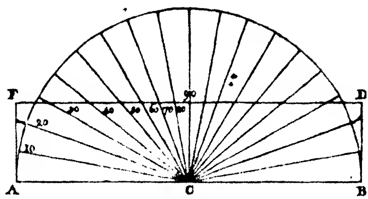
16. A *proposition*, in geometry, is a general term implying whatever is proposed to be effected, either by reasoning or by drawing lines, angles, &c. or by both. It may be either a problem or a theorem. A *problem* is something proposed to be done, and requires *construction*. A theorem is a truth stated, and requires *proof* or *demonstration*. A *corollary* is a property which evidently results from the demonstration of a proposition.

DESCRIPTION AND USE OF THE SEMICIRCLE, OR PROTRACTOR.

The protractor is a semicircle, generally made of brass, and divided into 180 equal parts or degrees, as in the annexed figure.



The degrees are sometimes transferred to the border of an ivory scale, in the following manner :



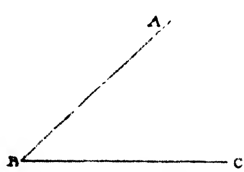
A semicircle is described with a radius equal to half the length of the scale A B. The semi-circumference is divided into 180 degrees, and radii being drawn from the centre C to each of the divisions, the points where they cut the borders of the scale are marked with the corresponding numbers.

PROBLEM I.—To make an angle containing any number of degrees.

Draw a straight line, and lay the centre of the protractor directly over the point from which the lines are to diverge, the diameter of the protractor being made to coincide with the line. Count the number of degrees along the circumference or border of the protractor, and make a small mark with the sharp point of the compasses. A line drawn from the centre of the protractor through that point, will form, with the other line, an angle containing the given number of degrees.

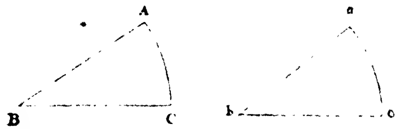
EXAMPLE.—Make angles of 25, 84, and 160 degrees.

PROBLEM II.—To find the number of degrees contained in a given angle A B C.



This being the *converse* of the last problem, the pupil is required to do it without assistance, and describe the method.

PROBLEM III.—To make an angle equal to a given angle A B C.

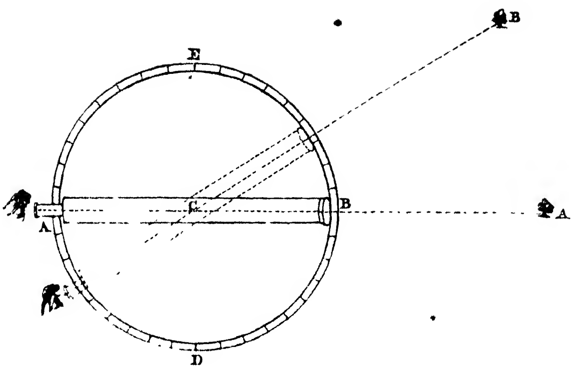


Draw a line *bc*. From B with any radius describe an arc cutting B C and B A, and from *b*, with the same radius, describe another arc cutting *b c* in *c*. From *c*, with a distance equal to C A, describe an arc cutting the former in *a*, join *b a*, and the angle *c b a* will obviously be equal to the angle C B A. (Why?)

DESCRIPTION AND USE OF THE THEODOLITE.

The theodolite is a circular instrument for the purpose of measuring angles contained between lines supposed to be drawn from any point to two distant objects. A very cheap and simple one for illustrating the principle of the instrument to pupils, and enabling them to ascertain the number of degrees in angles formed by distant objects, may be made in the following manner:

Let a circular piece of wood, eight or ten inches in diameter, be procured. Divide its circumference into 360 equal parts, and mark the degrees. This may be done more accurately if the surface have a sheet of strong white paper pasted or glued on it. A tube made of tin plate, in imitation of a small telescope, having a very small hole about the size of a pin in one end, and the other end having two fine wires soldered at right angles to each other, is made to turn round the centre of the wooden circle, so that it may be directed to a distant object. The instrument being placed on three feet of the proper height, the pupil will be able in a very few minutes to measure horizontal angles, and thus get a clear idea of *angular* magnitude.



The whole will be obvious from the annexed figure in which A D B E is the divided circle, and A B the tube in imitation of a telescope.

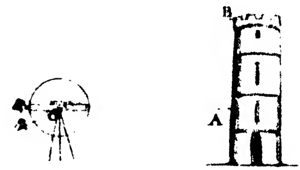
PROBLEM I.—To find the number of degrees contained in the angle formed by lines drawn from the centre of the theodolite to distant objects A, B.

Direct the tube to the object A, so that it may be opposite the point where the wires cross, and observe the degree or point where the tube crosses the circumference. Turn the telescope till the other object be seen opposite the intersection of the wires, and the number of degrees which the tube has passed over, will obviously be the number of degrees contained in the angle A C B.

If a glass tube nearly filled with water be fixed on the side of the tube A B, and parallel to it, a horizontal line may be ascertained. If the instrument be now placed on its stand with its plane in a vertical position, it may be used for taking angles in a vertical plane.

PROBLEM II.—To find the number of degrees contained in an angle formed between two lines, the one

horizontal, and the other supposed to be drawn to the top of an object.

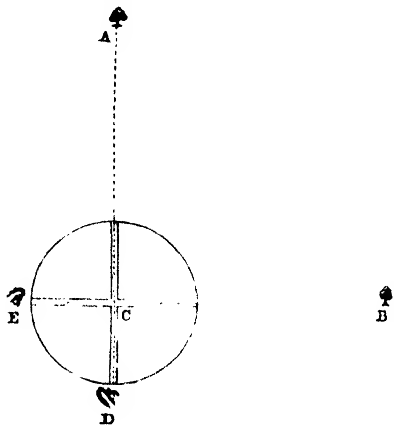


Fix the theodolite on its stand, place the tube in a horizontal position by means of the water level, and observe the degree on the circumference where the telescope crosses it. Elevate the tube till the top of the object be seen opposite the cross wires, and the number of degrees which the tube passes over will be the number of degrees contained in the angle A C B.

REMARK.—Unless the teacher be thus careful to give the pupil a clear idea of an angle by such contrivances, he will be apt to form a very erroneous idea of angular magnitude, and from the very derivation of the word, confound the idea of angular magnitude with that of a *corner*.

We would recommend the teacher to make his pupils take the angle formed by lines drawn to remarkable stars; also, to take the elevation of the north polar star, and thus get an approximation to the latitude of the place of observation. Before attempting to take the angle between two stars, or the altitude of the pole, the pupil might be allowed to practise with two candles placed at the extremity of the room, and also with one raised to a certain height, representing the polar star. The pupils will thus, at a very early period of their studies, begin to see a few of the uses of what they have learned; and delighted with the practical application of geometry, will go on vigorously with the study of a science which leads to such interesting results.

OBSERVATION. — There is a simple instrument, called a cross staff, which is very useful in determining the directions of two lines, on the field, which shall be at right angles to each other. It consists of a piece of wood about an inch thick, and about six inches in diameter,



having two narrow slits cut with a saw at right angles to each other, and about half an inch deep. On the other side, a hole is bored with a centre bit, to a sufficient depth for placing it on the top of a pointed rod, which may be pushed in the ground when the lines are to be determined.

USE.—To determine the position of one line C B, which shall be at right angles to C A. Place the circle horizontally on a rod or tripod. Place the eye at D, and turn it till you see the object A in the middle of the slit. Place the eye at E, and make another person plant a pole seen from E through the other slit. Then the angle A C B is obviously a right angle.

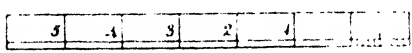
NOTE.—This simple instrument, if accurately constructed, will be found very useful in drawing lines at right angles to a given line on the field, when the perpendiculars are not very long.

A more accurate cross-staff may be made by soldering two

tin tubes, about nine inches long, at right angles to each other, each tube having a small hole in one end, and wires crossing each other at right angles in the other.

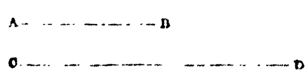
DESCRIPTION AND USE OF A SCALE OF EQUAL PARTS.

The pupil will find on one side of the scale contained in a case of instruments, a number of scales of equal parts; or he may make one for himself on a piece of smooth paper in the following manner:



Take any very short distance between the points of the compasses, and repeat it ten times along a straight line. Take the length of ten divisions, and repeat it along the line as often as may be necessary, and a scale of equal parts will thus be formed. By means of this scale, the lengths of lines may be compared with each other.

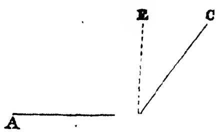
PROBLEM.—Compare the length of the line A B with C D.



Take the length of A B between the points of the compasses, and apply it to the scale of equal parts, and find how many of those parts are contained in it, suppose 20. Do the same with C D, and suppose it to contain 50. Then it is obvious that C D will be two and a half times the length of A B.

REMARK.—As this simple scale only gives the length of a line to two figures, the teacher may here explain the use of the diagonal scale, the description of which will be found in another part of the volume.

DEDUCTIONS FROM THIS SECTION.



1.—If there be three lines A B, B C, B D, drawn from the point B, having the sum of the two angles A B C, D B C, equal to two right angles, then A B D will obviously be a straight line.

For when A B D is a straight line, these angles amount to two right angles, therefore, conversely, when they amount to two right angles, A B D must be a straight line.

DEF.—The *supplement* of an angle is what it wants of two right angles.

Thus the angle A B C is the supplement of angle C B D, and C B D the supplement of A B C.

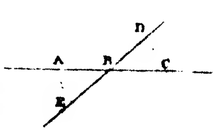
The *complement* of an angle is what it wants of a right angle. Thus, E B C is the complement of C B D.

EXAMPLES.—What is the supplement of an angle containing 25° 48′?

What is the complement of an angle containing 12° 15′?

2. If two straight lines cut each other, the opposite angles are equal to one another.

For the angles E B A and A B D are together equal to two right angles, and the angles C B D and A B D are also equal to two right angles, therefore the angle E B A is equal to C B D.

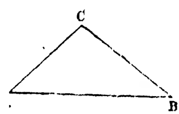


The same may be proved of the opposite obtuse angles.

SECTION II.

ON THE EQUALITY OF TRIANGLES, WITH A FEW PRACTICAL APPLICATIONS.

PROB. I.—It is required to make a triangle, whose base A B shall contain 36 parts from the scale of equal parts, A C 24, and the angle A 42°, also to find the length of the side B C, and the number of degrees in each of the angles at B and C.



Draw A B, A C, making an angle of 42°. Take from the scale a distance equal to 36 parts, and lay it off from A to B. Take a distance of 24 parts, and lay it off from A to C. Join C B, and A B C will obviously be the triangle required. Take the length of B C, and apply it to the line of equal parts, and its length will be found in numbers. The protractor being applied to the angles B, C, will give the number of degrees in each.

QUEST.—If you make another triangle in the same manner and with the same numbers, which of the triangles will be largest? The pupil will laugh at the simplicity of the question, and is sure to give the right answer.

The pupil may then cut them out of paper, and apply the one to the other, and he will clearly see that they not only *do* coincide, but they *must necessarily* do so. Hence the following theorem:

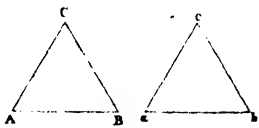
THEOREM.—If two triangles have two sides of the one respectively equal to two sides of the other, and the angles contained between those sides equal, the triangles shall be equal in every respect.

NOTE.—The pupil will use any of the scales of equal parts that may be most convenient for the size of his paper. For accurate measurement he should use as large a scale as pos-

sible, and draw the lines first very fine, with a point of the compasses.

Obs.—In order to see clearly the generality of the preceding problem and theorem, the pupil may construct a triangle, having its sides equal to any two lines, and the angle contained between them equal to any given angle.

Cor.—The angles at the base of an isosceles triangle are equal to each other.



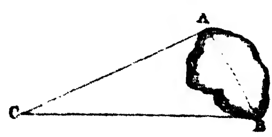
Let A C B, *a c b* be two isosceles triangles, having the sides A C, *a c*; C B, *c b*, all equal, and the angle C equal to *c*. If the triangle A C B be laid on *a c b*, the angle A will lie on *a*. If the triangle A C B be supposed reversed and laid on *a c b*, so that B C may now lie on *a c*, and A C on *b c*, then the angle B will now coincide with *a*. Hence, since each of the angles A, B have been shown to be equal to the angle *a*, they must be equal to each other.

Obs. 1.—If the pupil cut the equal triangles out of paper, and apply them to each other, he will clearly see that the angles A, B must be equal.

2.—This property, which is the 5th Prop. of Book 1. of "Euclid's Elements," has, on account of the difficulty of the demonstration, been called the "Pons Asinorum," or asses' bridge, because boys who are supposed to resemble that ill-used animal in certain respects, frequently fall over the bridge, and seldom recover from the fall.

PRACTICAL APPLICATION.

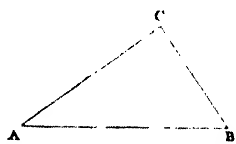
It is required to measure the breadth of a lake without crossing it.



Plant two poles perpendicularly in the ground at A and B. Place the theodolite at C, and find how many

degrees are contained in the B C A, suppose 25°. Measure with a tape the number of feet in C A, suppose 45, and in C B, suppose 52, then construct the triangle B C A, and measure the length of B A.

PROB. II.—Make a triangle whose base A B shall contain 46, the side A C 38, and B C 26 equal parts.



From the scale of equal parts take A B equal to 46. With a distance equal to 38 parts of the same scale describe an arc from the point A. Take a distance equal to 26, and from B, with that distance, describe another arc, cutting the former in C. Join A C, C B, and A C B will obviously be the triangle required.

QUEST. 1.—How many degrees are in each of the three angles?

2.—If another triangle be made in the same manner and with the same length of sides, which of them is the largest? Hence the following theorem:

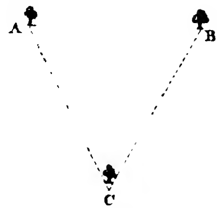
THEOREM.—If two triangles have all the sides of the one respectively equal to all the sides of the other, they are equal in every respect.

3.—Which are the angles that are equal to each other?

4.—Can you make a triangle whose base is 56, one of its sides 25, and the other 20? If not, why? Ans. Hence the following obvious deduction Any two sides of a triangle are together greater than the third side.

NOTE.—The pupil may generalise the problem and theorem as in the preceding problem. He may cut the triangles out of paper, and show that they *must* coincide.

PRACTICAL APPLICATION.



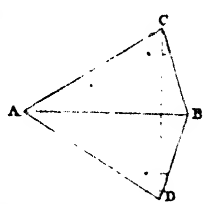
There are two trees A, B on a horizontal plane, and another at C, it is required to determine, without a theodolite, the angle A C B.

Measure the three sides, construct the triangle, and find the number of degrees in the angle C.

2.—How many degrees are in each of the angles of an equilateral triangle?

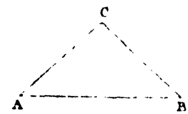
3.—Find, by construction and the protractor, the sum of the three angles of any triangle, whose sides are taken equal to any three lines.

OBS.—This theorem may be demonstrated without assuming that one circle can cut another only in two points.



Let A C B, A D B, be two triangles, having their sides respectively equal. Join C D, then in the isosceles triangle C B D, ∠ B C D = ∠ B D C. Also in the isosceles triangle C A D, ∠ A C D = ∠ A D C. Hence ∠ A C B = ∠ A D B, consequently the triangles A C B, A D B, have two sides, and the contained angles in each equal, and are consequently equal in every respect.

PROB. III.—It is required to make a triangle whose base A B shall contain 36 equal parts from the scale, and which shall have an angle at A of 48°, and at B of 24°. Required the lengths of of A C and B C, and the number of degrees in the angle C.



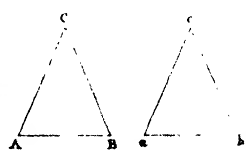
This problem is so like the preceding, that every pupil will be found able to construct it without the slightest assistance.

He may construct another in the same manner, and prove that they are equal by actually placing the one on the other. He may generalise it, and deduce the following theorem:

THEOREM.—If two triangles have a side of the one equal to a side of the other, and the angles at the extremities of those sides equal, the triangles will be equal in every respect.

QUEST.—Point out the equal angles. How do you know those angles that are equal to one another?

COR.—Hence, if a triangle have two equal angles, it will also have the sides opposite these angles equal.



For let A B C, $a\,b\,c$, be two triangles, having A B$=a\,b$; and the angles A, B, a, b, all equal, the triangles may be shown, as in Prob. I. to coincide in both positions: hence, A C being equal to $a\,c$ and C B equal to $a\,c$ A C is equal to B C.

PRACTICAL APPLICATION.

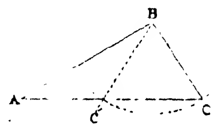
1.—To determine the breadth of a river without crossing it.

Fix on an object directly across the river, a tree for example. Place the theodolite at A, direct the telescope to C, and turn it round the centre till it has passed over 90°, and got to the position A B. Place a pole at B opposite the in-

tersection of the cross lines. Measure A B, place the theodolite at B, and measure the angle B. Suppose A B 50 feet and the angle B 35°, required the breadth of the river A C?

PROB. IV.—It is required to construct a triangle having the side A B equal to 34, the side B C 20, and the angle at A 30°.

Draw A B equal to 34, make an angle at A of 30°, and from B, with a radius of 20, describe an arc cutting the line A C in C and C′. Join B C, B C′, and A B C or A B C′ will be the triangle required.



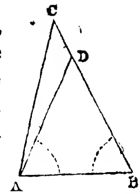
REMARK.—Since the arc cuts the line A C in two points, C and C′, it is obvious that from these data two separate triangles may be constructed, namely, A B C and A B C′; the one A B C an acute angled triangle, the other A C′ B an obtuse angled triangle. This may be shown to be the case from Prop. II. of the next Section. If the arc only touch the line A C, without cutting it, we shall have only one triangle.

If two other triangles be made in the same manner, each pair of the same kind will obviously be equal to each other. When they are like each other, they are by some authors said to be of the *same affection*. Hence the following theorem:

THEOREM.—If two triangles have two sides of the one respectively equal to two sides of the other, and the angle *opposite* one of the sides in the first equal to the angle opposite to the equal side in the second, these triangles are equal when they are of the same affection, that is, when they are both acute angled, both right angled, or both obtuse angled.

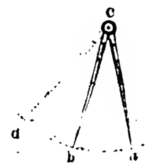
DEDUCTIONS AND EXERCISES FROM THE PRECEDING SECTIONS.

I.—The greater side of every triangle lies opposite the greater angle.



Let the angle B A C be greater than A B C, then B C is greater than A C. For if A D be drawn making ∠ B A D = ∠ A B C, then the two sides A D, D C of the triangle A D C will be together greater than A C; but B D is equal to A D, therefore B C is greater than A C.

Cor.—Conversely, the greater angle lies opposite the greater side.

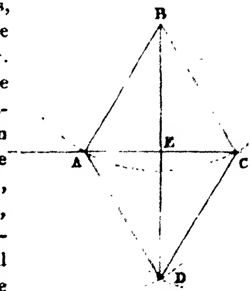


II.—Prove that the distance between the points of a pair of compasses increases by opening the legs, or increasing the angle *a c b*.

SECTION III.

ON THE BISECTION OF AN ANGLE, AND THE CONSEQUENCES RESULTING FROM THE PROBLEM.

Prob. I.—To bisect a given angle A B C.



From B, with any radius, describe an arc cutting the sides of the angle in A, C. From A and C, with the same distance, describe arcs cutting each other in D. Join B D, which will bisect the given angle. For join A D, C D, the triangles ABD, CBD, are equal.—(Why?) Therefore, the angle A B D is equal to the corresponding angle C B D.

IMPORTANT DEDUCTIONS FROM THIS PROBLEM.

If the triangle B C D be turned round B D, as on a hinge, (the pupil may cut out the triangle, and fold the paper along the line B D,) till it lie on B A D, then E C will lie on E A, and will therefore be equal to it; and the angle B E C will be on B E A, and each of them will therefore be a right angle, and the angle B C E will lie on B A E, and will consequently be equal to it. Hence the following problems and theorems are virtually contained in this problem.

I.—From a given point E, to draw a line perpendicular to a given line A C.

In A C take any two equal distances E A, E C, and from A and C, with the same distance, but greater than E A, describe arcs cutting each other in D or B. Join E, and the point of intersection, and B E or D E will be at right angles to A C, as has been already proved.

II.—From a given point, let fall a perpendicular on a given line.

From B, with a radius of sufficient length, describe an arc cutting the line in two points A and C. From these points, with the same radius, decribe arcs cutting each other in D, join B D, which will obviously be perpendicular to A C.

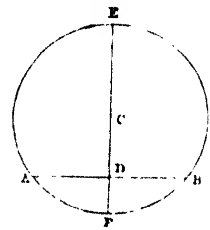
III.—To bisect a given line A C.

From A and C, with a radius greater than half the line, describe arcs cutting each other in B and D. The line joining B D will pass through the middle of A C, as has been already proved.

IV. Theorem.—Since B A = B C and since $\angle$ B A C has been proved equal to B C A, it follows that the angles at the base of an isosceles triangle are equal to each other, as has already been proved by a different process.

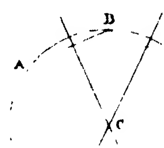
V. THEOREM.—Since A E C is the chord of an arc of a circle whose centre is B, it follows that if a straight line be drawn from the centre of a circle at right angles to a chord, it will bisect the chord. If it bisect the chord, it will be at right angles to it. If a line bisect a chord and be at right angles to it, that line will pass through the centre of the circle.

VI. PROB.—Given a circle to find its centre.



Draw any chord A B. Bisect it by the perpendicular E F, which will obviously pass through the centre. Bisect the diameter E F, and C will be the centre.

VII. PROB.—Given an arc to find the centre of the circle of which it is an arc.

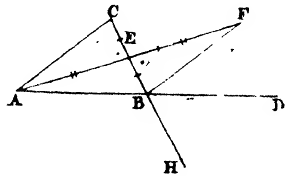


Draw any two chords A B, B D. Bisect each chord by a perpendicular, which will pass through the centre. Hence, their intersection C must be the centre.

VIII.—Describe a circle through three points A B C, which are not in the same straight line. The centre C will be found as in the preceding problem.

REMARK.—The pupil will readily perceive that all these propositions follow as obvious consequences from the general problem for the bisection of an angle. If the teacher think it necessary, he may make the pupil demonstrate each case separately.

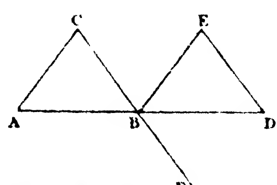
PROP. II. THEOREM.—If one side of a triangle be produced, the exterior angle C B D will be greater than either of the interior opposite angles A C B or B A C.



Bisect C B in E, join A E, and produce it till E F be equal to A E, join B F. Then the triangle A E C is equal to F E B, (why?) consequently $\angle$ A C E $=\angle$ E B F. But $\angle$ C B D is obviously greater than E B F, it is therefore greater than A C E. By bisecting A B, and going through the same process, it may also be shown that the angle C B D, or its equal A B H, is greater than B A C.

COR.—Any two angles of a triangle are together less than two right angles. For since $\angle$ C B D is greater than A C B, and since C B D and A B C are equal to two right angles, it follows that A C B and A B C must be together less than two right angles.

REMARK.—This property may be demonstrated in a different manner.



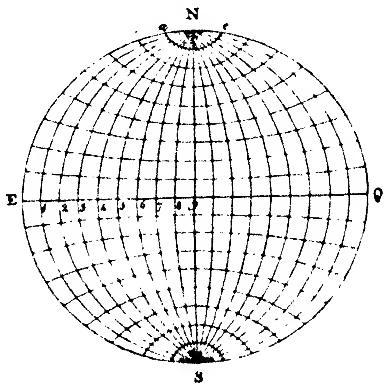
Let the triangle A B C be supposed to slide along the line A D till A coincides with B. It is obvious that the vertex of the triangle will now be to the right of the point C, and consequently B E must lie within the exterior angle C B D, and therefore be less than that angle. If C B be produced, the $\angle$ C may be shown in like manner to be less than the exterior angle A B F, or its equal C B D.*

* De Chale's Euclid.

IMPORTANT APPLICATION OF SOME OF THE PRECEDING PROPERTIES.

To construct a map of the world according to the globular projection.

In this construction the meridians are described at equal distances from each other, as are also the parallels of latitude.



Take any small distance and repeat it nine times along a line E 9. If E 9 be about the size which is wished for the radius of the map, the circle may be described, if not, a larger or smaller line may be taken till the diameter E Q be of the desired size. If it be required to make a map, whose radius must be of a certain length, it is only necessary to divide it into nine equal parts, which may be done by a few trials, or at once by Problem II. in Section V. At present, we shall suppose it may be made of any size, and consequently the radius will consist of nine small equal portions repeated along a line. With this radius, describe a circle. Draw a diameter N S at right angles to E Q, and repeat the same divisions along each of the radii. Divide each of the quadrants into nine equal parts. This may be done by drawing a line from the centre, making with the radius 9 Q, an angle of 10°. The arc intercepted between these lines, will be one-ninth of the quadrant Q N. This distance being repeated along each of the quadrants,

EXERCISES.

Ex. 1.—If an angle and its supplement be each bisected by a straight line, it is required to prove that the bisecting lines are at right angles to one another.

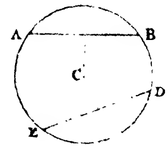
2.—Make an angle of 45° without the protractor.

3.—Prove that any two sides of a triangle are together greater than the third side.

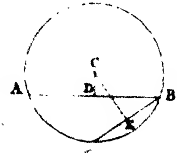
4.—Prove that the difference between any two sides of a triangle is less than the third side.

5.—Given two points, it is required to find, by means of the compasses only, any other point which shall be in the straight line joining them.

6.—Prove that equal chords are equally distant from the centre of the circle.



7.—It is required to prove that of two chords, the greater is nearest the centre of the circle.



SECTION IV.

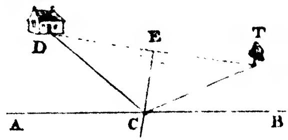
CONTAINING THE ELEMENTS OF GEOMETRICAL ANALYSIS, OR THE MODE TO BE PURSUED IN DISCOVERING THE SOLUTIONS OF GEOMETRICAL PROBLEMS.

1.—The usual mode of teaching geometry is to proceed step by step from what is already known till we arrive at what was unknown or required. This mode of procedure, which has been adopted in the preceding propositions, and which is adopted through the whole of the elements of Euclid, is called *Synthesis*. This term signifies combination or composition, because the result is obtained by combining in a proper manner the properties previously known. But though this method is well adapted for communicating the first principles of science to youth, it is a very feeble instrument when employed in the discovery of geometrical truth. A pupil who has studied the whole of the first Six Books of Euclid, according to the synthetical method, will have gained very little power in discovering the solution of a single problem which he has not previously seen, and which is not a mere deduction from a proposition which he has lately studied. Now pupils find infinitely more pleasure when they feel that they are every hour acquiring a power of intellect by which they can discover for themselves, than when they are merely committing to memory the discoveries of others. Very little progress is to be expected from a pupil before he begins to discover, and actually feel that " knowledge

is power;" and in geometry, one of the most useful and elegant of all the sciences, he can never feel this till he be able to wield geometrical analysis as an instrument of invention.

2.—Geometrical analysis is the reverse of synthesis. It consists in assuming that the problem has been actually solved. This is the point from which we start. We then observe the first consequence which naturally follows from that assumption, then another which necessarily follows from this, and so on, till we arrive at the conclusion that what was required is given. The moment this is accomplished, the analysis is complete. We have arrived at the discovery of the solution of the problem. If we now commence with the conclusion at which we have arrived, and retrace our steps to the point which was assumed, a complete synthetical solution of the problem will be obtained. We shall endeavour to illustrate the analytical and synthetical methods of investigation, by a few exercises requiring very simple combinations of the preceding propositions.

The pupil must bear in mind that the position of a point can only be found as follows: 1st. by a straight line cutting another straight line; 2ndly, by a straight line cutting the circumference of a circle; or, 3rdly, by the intersection of the arcs of two circles.



PROB. I.—Let D be the door of a cottage, T a tree, and A B a garden-wall—it is required to find a point C in the wall, which shall be at the same distance from the door of the house, that it is from the tree.

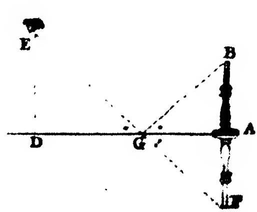
ANALYSIS.

Since the point C is supposed to be correctly found, C D will be equal to C T. Hence, if D T be joined, D C T will be an isosceles triangle. Hence, if D T be bisected in E, and E C joined, we shall obviously have two equal triangles, D E C, T E C, since D E is equal to E T, C D to C T, and C E common to both. Hence, the angle D E C is equal to T E C, or each of them is a right angle. But the line D T is given, therefore its middle point, E, is also given, and the line E C at right angles to D T is also given, and consequently its intersection with A B, or the point C, is also given. But C was the point required, the mode of finding which is now obvious.

SYNTHESIS.

Join D T, bisect D T in E, draw E C at right angles to D T, and the point of intersection of this line with A B will be the point required.

For since D E = E T, and E C common to the triangles D E C, T E C, and the angle D E C = T E C, these triangles are equal to each other. Hence the side D C is equal to T C, which was the thing required.



PROB. II.—Let A B be a candlestick with a candle, (the flame of which considered as a point, is at B,) standing perpendicularly on a plane mirror A D. Let E be the position of

the eye at the height of E D, above the mirror; it is required to find the point, G, at which the light is reflected which enters the eye, the angle E G D, or the direction of the angle which the ray of light after reflection makes with G D, being equal to A G B, the inclination with the mirror before reflection.

ANALYSIS.

Since the point G is supposed correctly found, the angle E G D is equal to A G B. Now, if E G, and B A, be produced to meet in F, we shall obviously have two equal triangles, B G A, F G A. For $\angle$ E G D $=\angle$ F G A, therefore, $\angle$ B G A $=\angle$ F G A and $\angle$ G A B $=\angle$ F A G, each of them being a right angle, and the side, A G, is common to both, consequently they are equal in every respect. Hence, A B $=$ A F. But A B is given, therefore A F is given, and consequently the point F is given, and the line E F, and the point of intersection G, which was the point required. Hence the mode of finding it is obviously the following.

SYNTHESIS.

Produce B A till A F be equal to it, join E F, and G will be the point required.

For since the triangle A G B is equal to A G F, (why?) the angle F G A is equal to A G B; but $\angle$ A G F $=\angle$ E G D, therefore, $\angle$ E G D $=\angle$ A G B. Hence G is the point required.

OBS.—Hence the reason why in a plane mirror the image F of the flame B appears as far behind the mirror as it actually is before it. Hence also the light has travelled through the shortest distance, being equal to the straight line E F, which is the shortest that can be drawn between two points.

These examples will be sufficient to point out the general mode of procedure in the analytical investigation of a geometrical problem. It is impossible to lay down a system of rules, which will, in every case, infallibly lead the pupil to the solution of the problem;

nor would it be judicious to give such rules, even if they did exist. For in that case the mental energies of the student could not be called into action; his inventive powers would lie dormant; geometry would lose its rank, as one of the most elegant and interesting of the intellectual sciences, and sink to the level of a mere mechanical art. But though definite rules cannot be laid down, a few practical observations may be useful in directing the pupil, generally, how to proceed.

1. He must first carefully examine the nature of the problem, and form a clear, well-defined idea of all the quantities concerned in the investigation.

2. He ought to construct as accurate a figure as possible, so that all the lines and angles may appear to the *eye* what they are in reality.

I would seriously advise the pupil, when he attempts the solution of a geometrical problem, to pay particular attention to this remark. If he construct his figure in a slovenly inaccurate manner, the *eye* will often mislead the *judgment*, and difficulties will be thus introduced which do not necessarily belong to the problem.

3. It will often be necessary to join certain points so as to form equal triangles, isosceles, or equilateral triangles, as in Problem I.

4. It will often be necessary to produce certain lines, to draw lines at right angles to others from certain points, to let fall perpendiculars from certain points on straight lines, as in Prob. II.

5. The pupil will often find it necessary to *bisect* straight lines; to bisect certain angles; to join one angle to another, so as to get the sum of two angles;

or to draw a line within another angle, so as to get the difference of two angles.

6. It will often be necessary to draw lines parallel to certain lines through remarkable points, which may be either given or required.

In short, the pupil must form such a combination of lines, angles, and circles, as will, in his judgment, lead to the discovery of the object required. If, after trial, he finds he cannot reach the point required, and take the citadel by the path he has sketched out, he must commence the attack anew by following a different road, and by adopting a different system of *tactics*. I would seriously urge a pupil not to be discouraged by failure in a first, second, third, or even fourth attempt. He is, with every new effort, even when unsuccessful, gaining additional strength of intellect and *personal* experience in the application of what he has learnt, and acquiring a habit of calm, patient investigation, which he will find more valuable through life, than all the knowledge that might be synthetically crammed into his head by the most diligent and laborious teacher. Pupils, in general, are too apt to run to their teacher for assistance at the first difficulty they encounter, and one of the most delicate parts of a teacher's duty is, to know what assistance he ought to give, so as to encourage a pupil to go on with the investigation, and find that he has, at least, accomplished a certain part of it himself. He will thus begin to feel the pleasure which results from actual discovery. When the difficulty, which is often a very trifling one, is removed *entirely* by the teacher on the first application, the pupil now sees that there was really no difficulty in the case, aud immediately exclaims, "How stupid I was not to have seen it!" A teacher

should carefully avoid calling a boy stupid, for if he do so, he will never hear him exclaim, "How stupid I was!" The moment I hear that exclamation from a pupil, I feel convinced that he will not be stupid long. His intellectual powers are now beginning to expand, and like the opening flowers, will unfold themselves with a rapidity proportional to the light which is shed over them. A boy will soon begin to be ashamed of frequently repeating this exclamation. He will now put his "shoulder to the wheel, and the whip to the horses," and will find, by experience, that he and his horses are generally able, *when perfectly willing*, to overcome most of the obstacles which at first seemed to stop their progress.

Particular illustrations of the preceding principles by reference to the analytical investigation of Prob. I.

Since the point C is supposed to be correctly found, we have D C equal to C T. But as we can deduce no conclusion from this without the comparison of triangles, we naturally join D T, and we have then an isosceles triangle D C T. Now, the only conclusion we can draw from this triangle, without additional construction, is, that the angle C D T is equal to C T D, but as neither of those angles are given, the position of the line D C, or T C, is not given, and consequently the point C is not given, and consequently cannot be found. Hence it is obvious that additional construction is required. The most obvious construction would therefore be to *bisect* D T in E, and join E C so as to form two equal triangles from which a certain inference might be drawn. The conclusion which obviously follows is, that E C is at right angles to D T, and its direction or position is given or determined, and consequently its intersection with A B is also *given*; but its intersection with A B, or the point C, was the point *required*. Hence the analysis is complete, or in other words, we have discovered the mode of determining the point C.

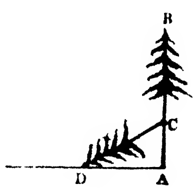
REMARK.—When the direction of a line is given without knowing its length, we say *it is given in position.* When its direction and length are both given, we say it is given in *position and magnitude.* When its length is given, without any reference to the direction in which it is drawn, we say it is *given in magnitude.*

QUEST.—Which are the lines in the two preceding figures that are given in position, and those that are given in position and magnitude?

OBS.—The pupil, ardent in the pursuit of useful knowledge, will now feel impatient to try his newly acquired strength; and will derive the same pleasure from the exercise of these powers, as the child does in the use of his limbs, when he makes the highly interesting discovery, that he can now walk without holding by his nurse's finger, or trusting to a leading string.

EXAMPLES FOR PRACTICE.

PROB. I.—Let A B represent a tree 80 feet high, standing on a horizontal plane, and let us suppose it to be broken by a wind at the point C, so that the part broken off may be thrown down in the position C D, the top touching the ground at D, 40 feet from the root. Required to construct the triangle D C A, and determine the height A C, where it was broken off?

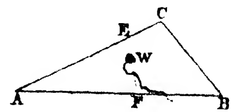


The pupil will try and find out the method by analysis, and then describe synthetically the construction with its demonstration.

II.—Given the distance A D, 40, as in the preceding figure, and the difference between C B, the part broken off, and A C, the part standing; 10 feet, it is required

to construct the triangle D A C, and find by means of the scale of equal parts the length of A C.

III.—Let A B C be a triangular field, and W, a well within it, it is required to divide the field into two parts by a line passing through W, so that the distances to A E, A F, shall be equal to one another. The mode of doing so to be found out analytically, and the construction and demonstration to be given synthetically.

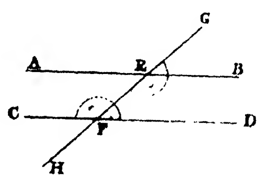


SECTION V.

ON THE PROPERTIES OF PARALLEL LINES.

PROP. I.—If a straight line cut two parallel lines, it makes the sum of the two interior angles on the same side of the line equal to two right angles.

Let the parallel lines A B, C D, be cut by G H. If the lines A B, C D, being produced towards the right, meet, they will form a triangle, and consequently the sum of the angles B E F, D F E, will be less than two right angles, and the sum of the two on the other side, viz. A E F, C F E, greater than two right angles. If the lines B A, D C, when produced towards the left, meet, they will form a triangle, and consequently the sum of the angles A E F, C F E, will be less than two right angles. Consequently when they do not meet in either direction, the angles A E F, C F E, must be together equal to two right angles.



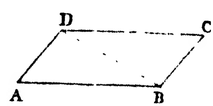
COR. 1.—Since the sum of the angles C F E, D F E is also equal to two right angles, it follows that $\angle$ A E F $= \angle$ D F E. These are called *alternate* angles. Hence a straight line cutting two parallel lines, makes the alternate angles equal.

2.—Since $\angle$ A E F $= \angle$ G E B, it follows that $\angle$ G E B $=$ D F E, that is, the exterior angle is equal to the interior opposite angle.

OBS.—Hence, if two lines be cut by a third so as to make the alternate angles equal, the exterior equal to the interior opposite angle, or the two interior on the same side of the cutting line equal to two right angles, these lines will obviously be parallel.

EX.—Prove that parallel lines are equally distant from each other in every point.

PROP. II.—If a four-sided figure have its opposite sides equal, they are also parallel.



For the triangles A B D, C D B, having all their sides equal, are equal in every respect. Hence the angle D B C $= \angle$ B D A, but these being alternate angles, A D is parallel to B C. Also D C is parallel to A B. (Why ?)

COR. 1.—Hence, if the opposite sides be parallel, they will be equal.

2.—Hence, also, if D C be equal and parallel to A B, A D may be shown to be equal and parallel to B C.

3.—Hence, the opposite angles of a parallelogram are equal.

4.—If one of the angles of a parallelogram or rhomboid be a right angle, all the angles are right angles.

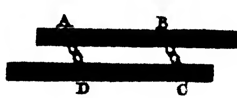
PRACTICAL APPLICATION OF THESE PROPERTIES.

I.—Let A B C represent a triangular piece of wood, having a right angle at B, with its edges straight and smooth for the purpose of drawing lines along them, a hole being cut in the middle for the purpose of holding it more easily. This instrument may be employed (it is so almost universally on the Continent) in drawing parallel lines and lines at right angles to one another.

1st. Through a given point A to draw a line parallel to a given line B C, by the triangle. (First fig. p. 36.)

Place it in the position A D C, one of its sides coinciding with the line B C, whilst the other passes through the given point. Draw the line D A E along its edge, slide it along A E, till its angular point comes to A, draw G A F along its edge, which will obviously be parallel to B C. (Why?)

Ex.—Draw a line parallel to D C by means of the *triangle*, founded on the property that the alternate angles are equal to one another.

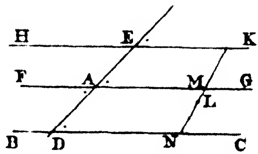


II.—The principle of the parallel ruler is to be found in Prop. II. From the very construction, it is obvious that A B, must in every position be parallel to D C. (Why?)

The pupil may now be allowed to use this instrument (practically) for drawing parallel lines.

Prob. I.—Through a given point A, to draw a line parallel to a given line by the scale and compasses.

Take any point D in the line, join A D, and through A draw F A G, making either of the angles F A D, or E A G, equal to the angle A D C, then F A G will obviously be parallel to B C. (Why?)



Ex. 1.—Prove that two lines F G, B C, which are parallel to the same line H K, are parallel to each other.

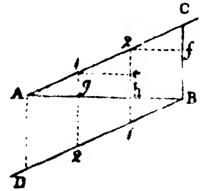
2.—From a given point A, it is required to draw a straight line A D, which shall make with a given line B C an angle A D C equal to a given angle (last fig.)

3.—Through a given point L, it is required to draw a straight line M N between two given parallels, so that M N shall be of a given length (last fig.)

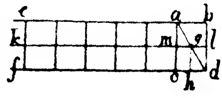
4.—If there be two angles having the sides of the one respectively parallel to those of the other, it is required to prove that these angles are equal.

PROB. II.—It is required to divide a given line A B into any number of equal parts, suppose three.

Draw any line A C, and B D, parallel to it. Take any distance, A 1, and repeat it three times along A C, repeat the same distance along B D. Join 1, 2; 1, 2 and A B will be divided into three equal parts in the points g, h. For join C B, A D, and draw 1 e, 2 f parallel to A B. Then it is obvious that the lines A D, 12, &c. are parallel.—(Why?) Consequently, the triangles A g 1, 1 e 2, are equal to each other.—(Why?) Hence A g=1 e, but 1 e=g h, hence A g=g h. In like manner, g h=h B.



CONSTRUCTION AND USE OF THE DIAGONAL SCALE.

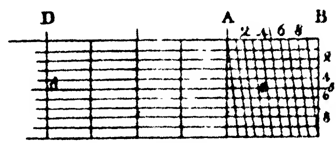


Let $e\,b$, $k\,l$, $f\,d$ be three equidistant parallel lines, having other equidistant lines drawn at right angles across them. Join $a\,d$, then $m\,g$ will be the half of $c\,d$ or $a\,b$. For draw $g\,h$ at right angles to $c\,d$, then the triangles $a\,m\,g$, $g\,h\,d$, are obviously equal, and hence $m\,g = h\,d = c\,h$; that is, $m\,g$ is the half of $c\,d$, or its equal $a\,b$.

If, instead of drawing one line $k\,l$ between $e\,b$, $f\,d$, there be drawn nine equidistant lines, the part $m\,g$ in the second line would obviously be $\frac{1}{10}$ of $a\,b$, the part on the next line $\frac{2}{10}$, &c.

Obs.—This scale is of extensive use in the accurate construction of geometrical figures.

Prob. I.—To take off from the scale a distance between the points of the compasses, which shall be represented by three figures, suppose 2 4 5.



If A B be called 100, then each of the divisions between A B will be 10, and the first distance between the perpendicular and diverging line below A will be $\frac{1}{10}$ of 10, or 1; between the next two divisions 2, &c. Hence the extent from d to e must be 245. If A B be called 10, each of the divisions between A B will be equal to 1, and the distances between the adjacent lines below A, $\frac{1}{10}$, $\frac{2}{10}$, &c. If A B be called 1, then each of the divisions between A B will be $\frac{1}{10}$, and the distances between the lines below A $\frac{1}{100}$, $\frac{2}{100}$, $\frac{3}{100}$, &c. Hence this scale is equally adapted for representing numbers by lines, whether these numbers be integers, decimal fractions, or whole numbers and decimal fractions.

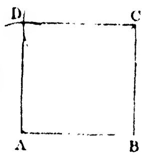
II.—To express the lengths of two or more lines by numbers.

Take the length, suppose equal to *d e*, and apply it to the scale, keeping the points of the compasses in the same horizontal line till the point *e* fall on the intersection of two lines, and the numbers representing the line will be determined. Thus in the preceding example, it will be either 245 or 24.5, or 2.45, &c. according as A B denotes 100, 10, or 1.

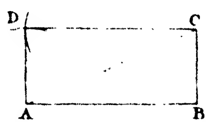
OBS.—The pupil may now employ the diagonal scale in the construction of a few simple figures, as in the following examples.

EXERCISES.

1.—Construct a square, A B C D, whose side shall be represented by the number 238. Required the length of the diagonal A C, in numbers?

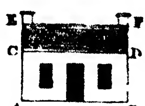


2.—Construct a rectangle, A B C D, A B being 246 and B C 124. Required the length of A C.

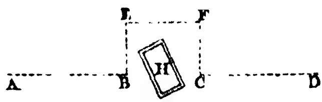


3.—Construct a parallelogram having the sides 128 and 86, and the angle contained between them 50°. Required the lengths of the diagonals.

4.—Construct the elevation of a cottage, the length being 20 feet, the height of the wall A C 7 feet, and of the roof 5 feet; the breadth of the door being 3 feet, and height 6 feet; each of the windows 2 feet broad, 3 feet in height, and 3 feet above the ground.



5.—Prove that no more trees can grow at the same distance from each other on a mountain, than on the horizontal base on which it stands.



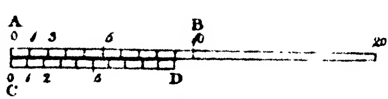
6.—It is required to continue the direction of a straight line, A B, beyond a house, H, or other obstacle, by means of the cross staff and tape, or chain.

7.—Prove that the diagonals of a parallelogram mutually bisect each other, and that in a square they cut each other at right angles.

8.—If two of the sides of a triangle be bisected, it is required to prove that the line joining the points of bisection is parallel to the third side, and equal to half of it.

9.—If all the sides of a quadrilateral figure be bisected, it is required to prove that the lines joining the points of bisection will form a parallelogram.

The property of parallel lines is thus extremely useful in getting small portions of a line without actual division. There is still another method of vast importance, and much greater generality in its application, with which the pupil should now be made acquainted.



Suppose an inch, A B, divided into 10 equal parts, each part will be $\frac{1}{10}$ of an inch. Again; suppose a line equal to 9 of these parts be also divided into 10 equal parts, each of those will be $\frac{1}{10}$ of $\frac{9}{10}$ of an inch, which is $\frac{9}{100}$. But the length of one of the first divisions being $\frac{1}{10}$, or $\frac{10}{100}$, and

that of the second $\frac{9}{100}$, one of the first divisions is $\frac{1}{100}$ of an inch longer than one of the second. If the line C D slide along parallel to A B till the two divisions marked 1, 1, form a continuous line, the sliding scale will have moved $\frac{1}{100}$ part of an inch towards B. If it slide along till the next two divisions coincide, it will have moved $\frac{2}{100}$ of an inch, &c.

To make this perfectly clear to a very young pupil, the teacher should have rules of various lengths, divided into equal parts, with sliding scales for each, so as to show the universality of the principle, and accustom the pupil to determine with accuracy the point of coincidence and the value of the divisions.

NOTE.—The sliding scale is called the Vernier, or Nonius, from the names of its inventors.

EXERCISES.

1.—If an inch be divided into eight equal parts, and seven of these parts be also divided into eight equal parts, how much is one of the first divisions longer than the second?

2.—If an inch be divided into 10 equal parts, and *eleven* of these parts be taken as the vernier, and divided into 10 equal parts, by how much is one of the last divisions longer than one of the first?

OBS.—Before applying this principle to the division of degrees into minutes, the pupil should be made fully and practically acquainted with some of its most useful applications. Take, for example, a barometer with its vernier, and let the pupil determine, not in a slovenly manner, but as accurately as the nature of the case admits, the height of the mercury in the tube above that in the cistern. If the inch be divided into ten equal parts, as was supposed in the illustration of the principle, the pupil has only to raise or depress the vernier till he see the cross-piece of brass directly opposite the surface of the mercury in the tube. He then looks for the point marked 0 on the vernier, and opposite this he sees the height in inches, and tenths of an inch nearly. He then observes the point where a division on the scale is in the same line with a division on the vernier, and the number marked at that point

on the vernier will give the additional part in hundredth parts of an inch. Thus, suppose 0 on the vernier stood a little above 29 inches and 4 tenths, and that the coincidence took place at the fifth division on the sliding scale, then the height of the mercury in the tube would be 29 inches and 4 tenths, and 5 hundredth parts, or $29\frac{45}{100}$, which is more commonly written 29.45 inches.

ADVICE.—We would seriously recommend to young gentlemen who have received what is called "a liberal education," to make themselves acquainted with the application of this principle, especially as a barometer is a very common piece of household furniture, and frequently looked at in a country where the state of the weather, and its effects on the nervous system, form a copious, interesting, and never-failing subject of conversation. We have heard of young men of a highly-finished education being placed in a very awkward situation by a lady asking the use of that little sliding scale which was always affixed to the best barometers.

APPLICATION OF THE VERNIER TO INSTRUMENTS FOR MEASURING ANGLES.

If 59 degrees on the circumference or *limb* of the instrument be divided into 60 equal parts, the difference between the length of one degree and one of the latter divisions will obviously be $\frac{1}{60}$ of a degree, or one minute. This kind of vernier, on account of its great length, is seldom employed. If each degree on the limb of the instrument be divided into two equal parts, or half degrees, and if 29 of these be taken and divided into 30 equal parts, the difference between the length of half a degree and one of the new divisions, will obviously be $\frac{1}{30}$ of half a degree, or $\frac{1}{60}$ of a degree, that is, one minute. This is a very common vernier. If each degree be divided into three equal parts, or third parts of a degree, and if 19 of those parts be divided into 20 equal parts, the difference between one of the first and one of the second division will be $\frac{1}{20}$, or the third part of a degree, that is $\frac{1}{60}$ of a degree, or 1 minute. This is also a very common vernier.

To use this scale, find the number of degrees, and halves, or thirds of a degree (according as the second or third scale is used), which the zero, or point marked 0 of the vernier, has passed over in taking an angle, find the point of coincidence on the vernier, which will give the additional minutes.

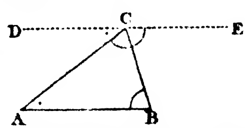
Thus, if the third kind of scale be employed, and if the zero of the vernier has passed over 15 degrees and 2 of the small divisions on the limb of the instrument, and if the coincidence takes place at 5 on the vernier, then the angle will be 15° 45′.

Remark.—A large wooden circle, like the horizon of a globe, with its accompanying vernier, should be procured, to exercise pupils in reading off degrees and minutes, before they can be entrusted with the use of a finer instrument.

SECTION VI.

ON THE SUM OF THE ANGLES OF TRIANGLES, AND, OTHER RECTILINEAR FIGURES.

Prop. I.—The sum of the angles of any triangle A B C, is equal to two right angles, or 180°.



Through C draw D E parallel to A B. The alternate angles marked with the same dots are equal. But the three angles at the point C are equal to two right angles. (Why ?) Hence the three angles of the triangle are equal to two right angles.

Cor. 1.—If the triangle be right angled the two acute angles amount to one right angle.

Remark.—A very simple popular demonstration may be given of this property. Suppose a ruler or straight line be applied to A B : if it be made to revolve about the point B, it will have acquired the position B C when it has
the angle B. Let it now revolve about C, it will have

the position C A when it has passed over the angle A C B. Let it now revolve about A; it will have arrived at the position A B when it has described the angle A, and will obviously have described half a complete revolution, or 180 degrees.*

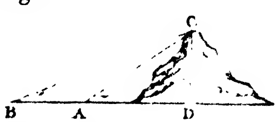
COR. 2.—If one side of a triangle be produced, the exterior angle will be equal to the sum of the two interior opposite angles; for, if one of its sides be produced, the *exterior* angle, together with its *interior adjacent* angle, being equal to two right angles, it follows that the exterior angle will be equal to the sum of the other two angles of the triangle, namely, the two *interior opposite angles.*

3.—Hence, if two triangles have two angles of the one respectively equal to two angles of the other, and a side *opposite* one of the angles in the one equal to the corresponding side in the other, they are equal. (Why?)

Ex. 1.—If two of the angles of a triangle be 54° 25′ and 62° 48′: required, the third?

2.—If one angle of a right angled triangle be 34° 15′: required the other acute angle?

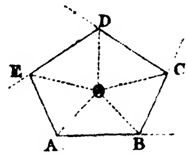
3.—Let B A be a horizontal plane, and C the top of a mountain; the theodolite being placed at A, the angle D A C is found to be 25°; and when removed in a direct line from C and A to B, the angle A B C is 15°, the distance A B being 568 feet. Required by construction with the protractor and diagonal scale, the perpendicular height, C D, of the mountain.



PROP. II.—The sum of all the interior angles of any rectilineal figure is equal to twice as many right an-

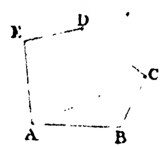
* Leslie's Geometry.

gles as the figure has sides, diminished by four right angles.



Take any point O within the figure, and from O draw lines to the angular points, thus dividing the figure into as many triangles as there are sides. The sum of all the angles of the triangles are equal to twice as many right angles as the figure has sides. But the sum of the angles round the point O, amounting to four right angles, not belonging to the angles of the figure, must be deducted from the result.

This may be demonstrated in a different manner.



From A draw the line A C, A D, and the figure will obviously be divided into as many triangles as it has sides wanting two triangles. For the first triangle, A B C, requires two sides of the figure, viz. A D, B C. The last triangle, A D E, requires also two sides of the figure, whereas each of the intermediate triangles requires only one side of the figure. Hence, if from the number of the sides we deduct two, the remainder will give us the number of triangles. Hence deduct two from the number of sides, and double this number, which will give the number of right angles equal to the sum of all the angles of the figure.

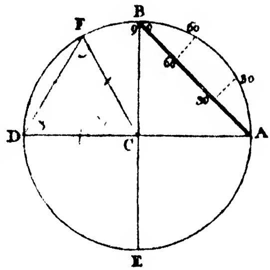
Ex. 1.—How many degrees are in each of the angles of a regular decagon, or ten-sided figure?

2.—If all the sides, A B, B C, C D, D E, E A of any rectilinear figure be produced, it is required to prove that all the exterior angles are equal to *four* right angles, whatever be the number of sides.

3.—If a figure have five equal sides and angles, how many degrees are in each of the angles?

APPLICATION OF THE PRECEDING PROPERTIES.

1.—Construction and use of the line of chords.

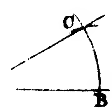


Let A B D E be a circle; draw two diameters, D A, B E, at right angles to one another; join A B, which will be the chord of 90°. If the arc A B be now supposed to be divided into 90 degrees, and straight lines drawn from A to each division, we should have the chords of all arcs, from 1 to 90 degrees. If the lengths of these chords be now transferred to A B, we shall have A B, the *line of chords*.

PROP. I.—THEOREM.—The chord of 60° is equal to the radius of the circle.

Let D F be the chord of 60°. Then in the triangle D C F the angle D C F is 60°; consequently, the sum of the other two is 120°; but these being equal to one another, each of them is 60°. Hence the triangle D C F is an equilateral triangle, and D F = D C.

PROP. II.—PROB.—To make an angle which shall contain a certain number of degrees, suppose 25°.



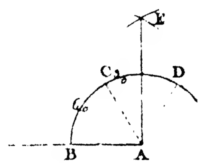
From A, with the chord of 60°, describe an arc, B C; take the length of the chord of the given number of degrees (25), and from B, with that distance, describe an arc, cutting the former in C, join A C, and B A C will be the angle required. For since B C is the chord of an arc of 25 degrees, the arc B C must contain 25 degrees, which is the measure of the angle A.

PROB. III.—To find the number of degrees contained in a given angle A.

This being the converse of the last problem, and being almost obvious, the pupil is required to describe the method of doing it.

IV.—To draw a line perpendicular to a given line from its extremity.

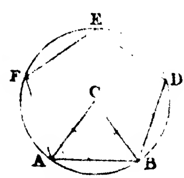
From A, with any radius, describe an arc, B C D; from B, with the same radius, cut the arc in C, and from C, with the same distance, cut it again in D; from C and D, with the same radius, describe arcs, cutting in E; join E A, which will be perpendicular to A B. Required the demonstration.



II. ON THE CONSTRUCTION OF REGULAR FIGURES BY MEANS OF THE SCALE, COMPASSES, AND PROTRACTOR, OR LINE OF CHORDS.

PROB. I.—On a given line A B, to construct a regular *pentagon*, or figure having five equal sides and angles.

The sum of all the angles $=2\times 5-4$ $=6$ right angles $=540°$; hence each angle contains 108 degrees; now lines drawn from the centre of the circle, passing through all the angular points, will obviously bisect each of the angles. (Why?) At A and B draw A C, B C making, with A B, angles of 54°. From C, with the distance C A, describe a circle, and A B repeated along the circumference will obviously go *five* times exactly; for the angle A C B will contain 72°, and consequently A B will be an arc of 72°, which is the fifth of 360°.



NOTE.—In like manner the learner may construct a regular hexagon, heptagon, octagon, nonagon, decagon, undecagon, and duodecagon.

PROB. II.—In a given circle to inscribe a regular polygon of any number of sides, a regular pentagon for example.

ANALYSIS.—(last fig.)

Suppose it done. Draw lines from the centre to the angular points; these angles will obviously be equal. Hence one of these, as A C B, contains the fifth part of 360° or 72°, consequently C A, C B, are given in position, and the points A B are given, and the side A B is also given.

SYNTHESIS.

Draw any radius C A, and another C B, making an angle A C B of 72°; then A B will go exactly five times round the circumference, and form the regular pentagon required.

EXERCISES.

Ex. 1.—In a given circle inscribe an equilateral triangle, hexagon, duodecagon, &c.

2.—Prove that patchwork may be formed by joining together *equilateral triangles*, *squares*, or *hexagons*.

3.—Prove that regular pentagons, or in short, any regular figures, except the preceding, cannot be employed in forming patchwork.

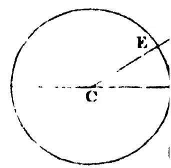
4.—Why did the bee select the regular hexagon for constructing its cell rather than the equilateral triangle or square?

5.—Prove that the perpendicular is the shortest line that can be drawn from a given point to a given line, and that the lines increase in length as they are more remote from the perpendicular.

SECTION VII.

PROPERTIES OF LINES AND ANGLES ABOUT THE CIRCLE.

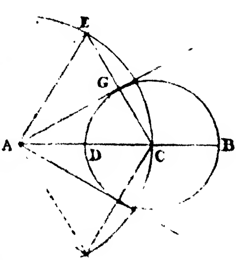
Theorem I.—A line drawn at right angles to the radius at its extremity, is a *tangent* to the circle, or line touching the circle at A.



Take any point D in the line, join C D, then since C D is longer than C A, (why?) it is longer than C E, and consequently D is without the circle. Hence A D is a tangent.

Ex.—Draw a tangent to a circle from a point A in the circumference.

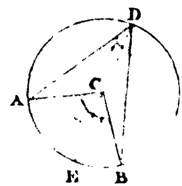
Prob.—To draw a tangent to a circle from a point A without it.



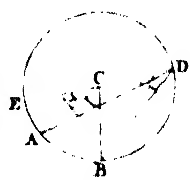
Draw A B through the centre. From A with the radius A C, describe an arc, from C, with a distance equal to D B, cut the former in E, join C E cutting the circle in G, and G will be the point of contact. For since A C is equal to A E, and C G to G E, (why?) and A G common to both, the triangles A G C, A G E, are equal, and the angle A G C is equal to A G E, or A G C is a right angle, and consequently A G is a tangent to the circle. The same construction will give the tangent on the other side of A B.

Cor.—Hence, the tangents to a circle from the same point are equal.

Theorem II.—The angle at the centre of a circle is double the angle at the circumference, both standing on the same arc.



Case 1.—Join D C, and produce it to E, then the exterior angle A C E of the isosceles triangle, A C D, being equal to the interior opposite angles, is double one of them, namely, A D C; for the same reason B C E is double B D C. Hence the whole angle A C B is double the angle A D B.



2.—Join D C and produce it to E. Then the exterior angle E C B of the isosceles triangle B C D is double of the interior opposite angle C D B or E D B. In like manner E C A is double of C D A; hence the difference of the angles E C B, E C A, viz. A C B, must be double the difference of the angles E D B, E D A, viz A D B.

Note.—The pupil will often find it useful to arrange his demonstrations in the form of equations, thus:

$$\angle ECB = \angle CDB + \angle CDB$$
$$\angle ECA = \angle CDA + \angle CDA$$

By subtraction $\angle ACB = \angle ADB + \angle ADB =$ twice ADB.

Should he still find a difficulty, he may take the quantities in numbers, thus:

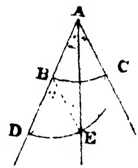
$$12 = 6 + 6$$
$$8 = 4 + 4$$
$$4 = 2 + 2 = 2 \times 2.$$

Cor. 1.—Hence all angles at the circumference which stand on the same arc are equal to one another, since each of them is half the angle at the centre. An angle at the circumference has for its measure half the arc on which it stands.

Obs.—In a circular amphitheatre, the spectators will see the front of the stage under the same angle in whatever part of the *circumference* they may be placed.

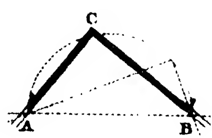
Cor. 2.—This property may be employed in bisecting an angle.

From the angular point A, with any radius, describe an arc B C, from B with the same radius describe the arc D E. Take the chord of B C, and lay it off from D to E, join A E, which will bisect the given angle B A C. Required the demonstration.



Cor. 3.—Hence an arc of a circle may be described by means of two straight pieces of wood fixed at a certain angle.

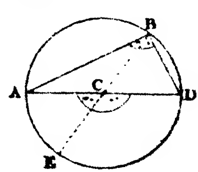
Fix two pins A, B, perpendicularly in the surface on which the arc is to be described. Move the angular instrument round, keeping the inner sides pressing against the pins, and a tracing point or pencil at C will trace out an arc of a circle.



Remark.—This method may be employed in describing the meridians and parallels of latitude in a map of the world on a large scale.

Cor. 4.—An angle in a semicircle, or standing on half the circumference, is a right angle.

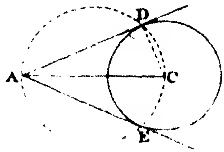
For the angle A B D is made up of the angles A B E and D B E, which are respectively the halves of the angles A C E and D C E; but the sum of the last two is two right angles, therefore the sum of the first two, or A B D, is one right angle.



EXAMPLES.

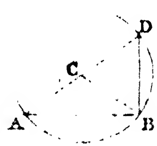
1.—From this property to draw a tangent to a circle from a point A without it.

Join A C, and on A C describe a semicircle cutting the circle in D, draw A D, which will obviously be a tangent to the circle. (Why?)



2.—It is required by means of this property to draw a perpendicular to a given line A B from its extremity.

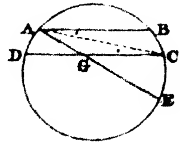
Take any point C above A B, and from C with the radius C B, describe an arc cutting the line in A, join A C and produce it to meet the arc in D, join D B, which will obviously be at right angles to A B. (Why?)



3.—By the application of the same property, to let fall from a given point D a perpendicular to a given line.

PROP.—If two parallel chords be drawn in a circle, they will intercept equal arcs.

For the angle B A C = A C D, therefore the arc A D = the arc B C.



Ex.—Through a given point A, draw A B parallel to D C by this property.

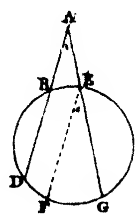
PROP. III.—If two straight lines cut one another within a circle, the angle which they form is equal to an angle at the centre which stands on an arc equal to half the sum of the included arcs.

Let A E, D C, (see last fig.) be the lines cutting in G. Draw A B parallel to C D, then the arc B C is equal to A D. Hence the arc B E is the sum of E C, and A D. But the angle E G C is equal to B A E, and B A E is equal to an angle at the centre standing on half the arc B E.

REMARK.—This property is of use in getting the true angle by means of a theodolite not properly *centered*. It may also be employed in measuring an angle on paper by means of a circular protractor.

Ex.—If the arc B E on one side of the instrument be 48° 20′, and the corresponding arc A D on the opposite side be 20° 18′, the instrument not being properly centred, required the true angle.

Ex.—If two lines cut each other without a circle in A, it is required to prove that the angle D A G is equal to an angle at the circumference, standing on an arc equal to the difference between B E and D G.

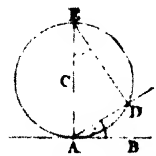


This property may also be employed for measuring an angle on paper, or drawing lines, forming any angle by means of a circular protractor.

The pupil may be allowed to make such a protractor on a piece of strong paper, and show its practical application in both (

Ex.—If the arc B E contain 40° and D G 100°, how many degrees are contained in the angle D A G?

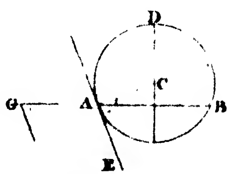
Theorem IV.—If a tangent be drawn to a circle, and from the point of contact a line be drawn, cutting off an arc, the angle between the tangent and the *secant* will be equal to an angle at the circumference of the circle standing on the arc cut off.



Let A B be a tangent, and A D a line cutting off the arc A D; then the angle B A D is equal to an angle at the circumference standing on A D. For, draw A E at right angles to A B, and join E D. Then since A D E is a right angle, the two angles A E D and E A D amount to a right angle; but B A D and E A D are also equal to a right angle, therefore $\angle$ B A D = $\angle$ A E D.

EXERCISES.

1.—On a given line, A B, to describe an arc of a circle which shall contain an angle equal to a given angle G.

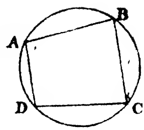


Draw A E, making the angle B A E equal to G. Bisect A B by a perpendicular; draw A C at right angles to A E, meeting the bisecting line in C; from C, with the radius C A, describe a circle, and the arc A D B will contain an angle equal to G. Required the demonstration.

2.—From a given circle to cut off an arc, A D B, which shall contain any number of degrees, suppose 35.

3.—From a given point, A, in the circumference of a circle, to draw a tangent, A E, without finding the centre of the circle.

Prop. V.—In any quadrilateral figure inscribed in a circle, the sum of the two opposite angles is equal to two right angles.



For the angle B A D has for its measure half the arc B C D, and the opposite angle, B C D, has for its measure half the arc B A D. Hence, the measure of the sum of those angles is half the circumference, or 180°. The same reasoning applies to the angles B, D.

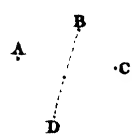
EXERCISES.

Ex. 1.—It is required to prove that two circles can cut each other only in two points.

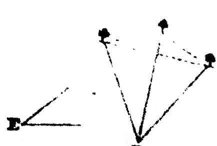
2.—It is required to prove that the longest line that can be drawn to the opposite concave side of a circle from a point without it, is that which passes through the centre.

3.—It is required to prove that the shortest line which can be drawn from a point without a circle to the convex circumference, is that which, being produced, passes through the centre.

4.—If two circles touch one another, it is required to prove that the centres and the point of contact are in the same straight line.

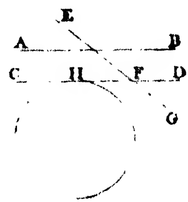


5.—Given, the position of three points A, B, C, not in the same straight line; it is required to find the point D, so that B D shall be the diameter of the circle passing through A B C. The point D to be determined without finding the centre of the circle.



6.—Given, the position of three objects; it is required to find the point B, so that the two adjacent objects shall be seen under the same angle, the

angle under which each pair is to be seen being equal to a given angle E.



7.—It is required to draw a tangent to a circle which shall be parallel to a given line A B.

8.—It is required to draw a tangent to a given circle, which shall cut a line E G given in position at a given angle, suppose 20°.

9.—It is required to find the centre of a circle which shall touch a given line, C D, in a given point, H, and shall pass through another given point.

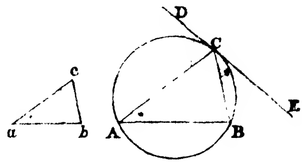
SECTION VIII.

ON THE SIMPLEST KINDS OF INSCRIBED AND CIRCUMSCRIBING FIGURES, AND THE CONSTRUCTION OF REGULAR FIGURES.

DEF.—If a circle pass through the angular points of a rectilineal figure, it is called the *circumscribing circle*, and the figure is called the *inscribed* figure. If a circle touch all the sides of a rectilineal figure, it is called the *inscribed circle*, and the figure itself, the *circumscribing figure*.

PROP. I.—In a given circle to inscribe a triangle which shall have its angles respectively equal to those of a given triangle.

ANALYSIS.



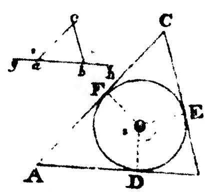
Let the triangle A B C, having its angles respectively equal to *a*, *b*, *c*, be supposed inscribed in the given circle. Through C draw a tangent to the circle. Then since the angle B C E is equal to A, which is equal to *a*, the line C B is given in position, and consequently the point B is given. In like manner, the point A is given, (why?) Hence the three points C B A are given, and consequently the triangle A B C.

SYNTHESIS.

Take any point C in the circumference, and at that point draw a tangent. Draw C B, making the angle B C E equal to the angle *a*, and C A making the angle D C A equal to the angle *b*, join A B, and A B C will be the triangle required: for the angle B will be equal to *b*, and A to *a*, and consequently A C B to *c*. (Why?)

PROP. II.—About a given circle to circumscribe a triangle, which shall have its angles respectively equal to those of a given triangle.

ANALYSIS.



Let the triangle A B C be supposed described about the given circle, so that its angles shall be respectively equal to those of the given triangle *a b c*. Join the centre O and the points of contact D, E, F. In the figure D O E B, the angles D O E and D B E amount to two right angles, (why?) But the angle B being given, its supplement is also given, hence the two radii O D, O E, are given in position. In like manner

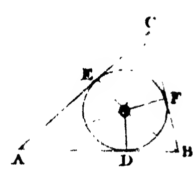
the angle D O F may be shown to be given, and consequently the line O F is given in position. Hence the intersections of those lines with the circumference, or the points D E F are given, and consequently the tangents A B, B C, A C are given in position, and consequently the triangle A B C is given.

SYNTHESIS.

Produce the base of the given triangle. Draw any radius O D, and another radius O E, making the angle D O E equal to the exterior angle *c b h*. Draw O F, making the angle D O F equal to the angle *c a g*; at the points D, E, F, draw tangents, and these tangents, being produced to meet, will form a triangle A B C, having its angles A, B, C respectively equal to *a*, *b*, *c*. (Why?)

Prop. III.—In a given triangle to inscribe a circle.

ANALYSIS.



Let A B C be the given triangle, and let the circle E D F be conceived to be inscribed. Join A O, B O, C O. Since O E = D O, A O common to the triangles A D O, A E O, and the angles at E, D right angles, these triangles are equal, consequently ∠ D A O is equal to O A E, and therefore A O is given in position. In the same manner B O may be shown to be given in position, and consequently their intersection O is also given.

SYNTHESIS.

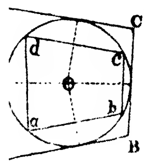
Bisect any two of the angles. The point of intersection of these lines will be the centre of the inscribed circle, and its radius the perpendicular from that point on one of the sides. Required the demonstration.

Cor.—Hence, if the three angles of a triangle be bisected by straight lines, these lines will meet in the same point.

Ex.—To circumscribe a circle about a given triangle.

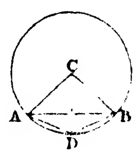
Prop. IV.—Having given any rectilineal figure inscribed in a circle, it is required to construct a circumscribing figure, having its angles respectively equal to those of the inscribed figure.

From the centre O let fall perpendiculars on all the sides, and produce them to meet the circumference, at those points draw tangents, and these tangents will form a figure A B C D, which will obviously have its angles A B C D respectively equal to *a b c d*. (Why?)



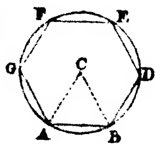
Prop. V.—Given a regular inscribed figure, it is required to inscribe another, having twice the number of sides.

Let A B be the side of a regular figure inscribed in a circle. Bisect the angle A C B by C D, join A D, which will be the side of the required figure. For the triangles A C D, B C D are equal, and consequently A D = B D. Hence the side A B is exchanged for the two sides A D, B D.



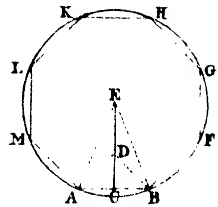
Prop. VI.—On a given line, A B, to describe a regular hexagon without the line of chords or protractor.

On A B describe the equilateral triangle A C B, from C, with the radius C A describe a circle, then A B will go exactly *six* times round the circumference, and form a regular hexagon A B D E F G. Required the proof.



Prop. VII.—On a given line, A B, to describe a regular *octagon*, or figure having eight equal sides.

Bisect A B by the perpendicular C D. Make C D equal to A C, and join A D. Make D E equal to A D, and E will be the centre of the circle which will circumscribe the required octagon. From E with the radius A E describe a circle, repeat A B along the circumference, and A B F G H K L M will be the octagon required. For A D B is a right angle, (why?) and A E B is half a right angle, (why?) Therefore eight angles equal to A E B will fill up the space round the point E, or the chord A B will go eight times round the circumference.



REMARK:—The investigations of the regular pentagon and decagon will be given in the Second Part of the Volume.

EXERCISES.

1—In a given circle to inscribe a square and a regular octagon.

2.—About a given circle to describe a square and a regular octagon.

3.—In a given square to inscribe a circle.

4.—About a given square to describe a circle.

5.—In and about any regular figure to describe a circle, and to show that the centres of the inscribed and circumscribing circles are the same.

6.—In a given circle to inscribe an equilateral triangle, hexagon, and duodecagon.

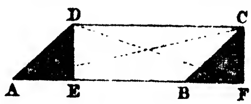
7.—If a regular polygon be inscribed in a circle, it is required to prove that if tangents be drawn at the angular points, the figure formed by the tangents will be a regular circumscribing figure, having the same number of sides.

SECTION IX.

ON THE EQUALITY OF SURFACES.

Prob.—To change the form of a parallelogram without altering the surface.

Let A B C D be the parallelogram. Cut off any triangle A E D, and place it in the position B F C and E F C D will be the parallelogram required.



Note.—The pupil may cut the figures out of paper, and prove that E F C D is a parallelogram. By cutting off any other triangle and sliding it along to the other side, the length of the parallelogram may be increased to any extent.

Cor. 1.—Hence parallelograms standing on the same or equal bases, and between the same parallels. or having the same altitude, are equal.

2.—Since the triangles A B C, E F D, are the halves of equal parallelograms, it follows, that triangles standing on the same or equal bases, and between the same parallels, or having equal altitudes, are equal to each other.

3.—If a triangle and parallelogram stand on equal bases, and have the same altitude, the parallelogram is double of the triangle. If the base of the triangle be double that of the parallelogram, and the altitude the same in both, they are equal.

4.—Conversely. Equal parallelograms or on the same or equal bases, have equal altitudes,

EXERCISES FOR CONSTRUCTION.

1.—Convert a triangle into an equal rectangle.

2.—Change a given triangle into an equal parallelogram, which shall have one of its angles equal to a given angle, suppose an angle of 35°.

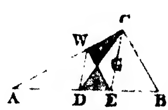
3.—Convert a given parallelogram into an isosceles triangle.

4.—Divide a triangle into two, three, or any number of equal triangles, by lines drawn from the vertex to points, which are to be found by construction, in the base.

APPLICATION OF THE PRECEDING PROPERTIES.

PROB. I.—There is a triangular piece of ground having a well at W, it is required to divide it into two equal parts, so that each may have communication with the well.

ANALYSIS.

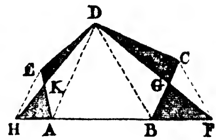


Let W E be the division required. Bisect A B in D, and join C D. Join also W D and C E. Then, since the triangle A D C is half the field, it is equal to A W E, and therefore the small triangle C W G is equal to D E G, and consequently the triangle C W D is equal to D E W, (why?) But these triangles stand on the same base W D, they must therefore have the same altitude, or be between the same parallels. Hence C E is parallel to W D. But W D being given, C E is given in position, consequently its intersection E with the base is also given; and therefore W E, the line required, is also given.

SYNTHESIS.

Bisect A B in D, join W D, and through C draw C E parallel to W D, cutting the base in E; join W E, and prove that the triangle A W E is equal to A C D, or half the triangle A B C.

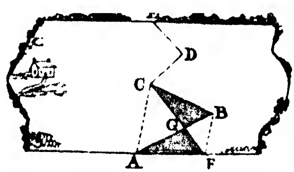
II.—To change any rectilineal figure, A B C D E, into a triangle having the same area.



Join B D, and draw C F parallel to it, join D F, and the five-sided figure will be converted into the equal four-sided figure A E D F. For the triangle D G C, which is left out, is obviously equal to G B F, which is taken in. By the same process the figure may be reduced to an equal figure having one side less, and so on till we arrive at a triangle H D F, equal to the original figure.

The triangle may now be converted into a rectangle.

III.—To convert a crooked boundary A B C D E between two farms into a straight one without changing the size of the farms.



Join A C, and draw B F parallel to it, join C F and the boundary will be changed into E D C F, for the triangle C B G taken from the west farm and added to the east, is by the last Proposition, equal to the triangle A G F taken from the east and given to the west farm. By joining F D and going through the same process, a new boundary will be found having one bending less, and so on till the crooked boundary be reduced to a straight line.

EXERCISES.

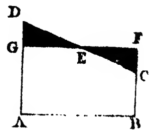
1.—Convert any rectilineal figure into a rectangle.

2.—Convert any crooked boundary between two farms into a straight one without altering the respective size of the farms.

Prop. II.—To convert a trapezoid into an equal parallelogram.

Def.—A trapezoid is a four-sided figure, having two of its sides parallel but not equal.

Let A B C D be a trapezoid. Bisect D C in E. Through E draw G F parallel to A B meeting B C produced in F. Then the parallelogram A F will be equal to the trapezoid.



For triangle D E G = C E F.

Hence a trapezoid may be changed into an equal parallelogram by retaining one of its sides A B, and taking a line A G, equal to half the sum of the parallel sides for the other side of the new parallelogram.

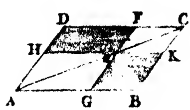
EXERCISES.

1.—Prove that A G is half the sum of the parallel sides B C, A D.

2.—If A B be 10 feet, A D 16, and B C 12, what are the sides of an equal parallelogram?

Prop. III.—The complements of the parallelograms about the diagonal of a parallelogram, are equal to each other.

Def.—Let H G, F K be parallelograms having A E, E C, for their diagonals, and their sides parallel to those of the parallelogram A B C D, then the parallelograms D E, E B are called the complements of the two parallelograms, or the portions which are required to fill up the parallelogram D B.



Demonstration.—Triangle A B C = A D C.
Triangle A G E = A H E.
Triangle K E C = F E C.

Therefore parallelogram E B = parallelogram E D.

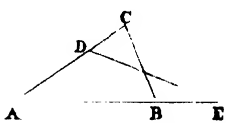
Cor.—Hence if there be any parallelogram D E, another E B, may easily be found equal to it, and which shall have one of its sides E K equal to a given line.

Ex.—Given a parallelogram H E F D, having H E 324 feet, and H D 136, and the angle E H D 40°, it is required to construct a new parallelogram E B equal to the former, having one of its angles = 40°, and one of its sides E K equal to 214 feet, and to give by accurate construction the number of feet in E G.

EXERCISES ON THE PRECEDING PROPERTIES.

1.—From a given point in one of the sides of a parallelogram, it is required to draw a line to the opposite side, dividing the figure into two equal parts.

2.—It is required to change the triangle A B C into an equal triangle A D E, which shall have its base on A B produced and its vertex in a given point D. The investigation to be conducted by geometrical analysis.



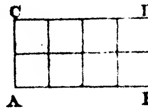
PRACTICAL APPLICATION OF THE PRECEDING PROPOSITIONS TO DETERMINE THE NUMERICAL RELATIONS OF SURFACES BOUNDED BY STRAIGHT LINES.

In the comparison of lines of different lengths we employ a certain measure, which is called a unit, as for example, an *inch*, a *foot*, a *yard*, &c. In comparing the surfaces of figures we employ a square described on the lineal unit of measure, which is called a *square inch*, a *square foot*, a *square yard*, &c.

The mensuration of surfaces consists in ascertaining how many squares of a certain size are contained in the surface of a given figure. The *area* of a figure is

the number of squares of a certain unit which it contains.

Prob. I.—To find how many square units are contained in a given rectangle.



Let A B contain four units, and A C two; then, by drawing the parallel lines as in the figure, it is obvious that the rectangle is divided into eight squares, which number is obviously found by multiplying the length by the breadth. Now, as any parallelogram is equal to a rectangle having the same base and altitude, we have the following rule for finding the area of any parallelogram.

Rule.—Multiply the number of lineal units in the length by the perpendicular breadth.

1.—How many square inches are in a square foot? How many square feet in a square yard?

2.—How many square yards in a rectangular garden, the length being 420 feet and breadth 240?

Prob. II.—To find the area of a triangle when the base and perpendicular are given.

Rule.—Multiply the base by half the perpendicular, or the perpendicular by half the base.

Ex.—How many square yards are contained in a triangular field whose base is 860 feet and perpendicular 480?

Prob. III.—To find the area of a trapezoid.

Rule.—Multiply half the sum of the parallel sides by the perpendicular distance between them.

Ex.—How many square feet are contained in a deal whose length is 20 feet, the breadth of one end being 18 inches and of the other 10?

NOTE.—These rules are obvious deductions from the properties demonstrated in the last Section. The pupil is required to trace them to their source.

2.—The rectangle contained by two straight lines is expressed by the product of the lines; thus, in the preceding figure the rectangle contained by the lines A B, B D, that is, the area of the rectangle A B D C is expressed by $AB \times BD$ or $AB.BD$. When A B is equal to B D, or when the rectangle becomes a square, its area is expressed by placing the figure 2 above the line denoting the length of its side; thus the square described on A B may be expressed by $AB \times AB$. or AB^2.

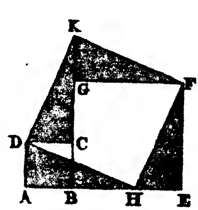
The square described on the sum of two lines A B, B D, is expressed thus $(AB + BD)^2$ or $\overline{AB + BD}$, and on their difference thus $(AB - BD)^2$ or $\overline{AB - BD}$; and the rectangle contained by their sum and difference is expressed thus,

$$(AB + BD)\,(AB - BD) \text{ or } \overline{AB + BD} \times \overline{AB - BD}.$$

SECTION X.

ON THE 47TH PROPOSITION OF THE FIRST BOOK OF EUCLID'S ELEMENTS, WITH SOME OF ITS PRACTICAL APPLICATIONS.

PROP. I.—PROBLEM.—Given two squares; it is required to cut them in such a manner as to form a new square, with the segments into which the two squares are divided.



Let the squares A B C D, B E F G, be placed adjacent to one another as in the figure; take the length of the side A B of the smaller square, lay it off from E to H, and join H F, H D; cut off the triangle D A H, and place it in the position D C K; cut off F E H, and place it in the position F G K, and

it will obviously fill up that space. (Why?) The figure D H F K will be the square required.

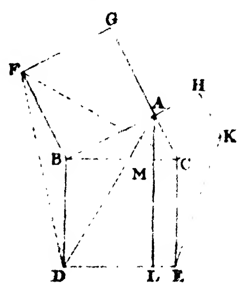
For the triangle D A H being equal to H E F, (why?) the figure D H F K has all its sides equal. Again; the angle E F H is equal to the angle A H D; but E F H and E H F amount to one right angle, consequently the two angles A H D and E H F amount to one right angle. Hence the angle D H F is a right angle, and consequently D H F K is a square.

COR.—Since H E was made equal to A B, and since the square D H F K is equal to the square described on E F, together with the square described on H E, the following theorem follows as a necessary consequence: —*The square described on the hypothenuse, or side opposite the right angle, of a right-angled triangle is equal to the sum of the squares described on the other two sides.*

OBS.—As this is one of the most important properties of the triangle, mathematicians seem to have exhausted their ingenuity in devising different modes of demonstration. The most important or elegant of these we shall now lay before the pupil.

DEMONSTRATION I.

Let B A C be the right-angled triangle, and B G, C H, C D, the squares described on its sides; then the square C D will be equal to the squares B G, C H.



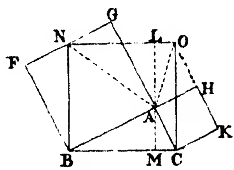
1.—Join F C, A D, then the triangles, A B D, F B C are equal, because they have two equal sides, and the angles contained between those sides equal. (Name the equal sides and angles.)

2.—The triangle F B C is half the square B G, because it

stands on the same base, F B, and between the same parallels, F B, G C. (Why is G C a straight line?) The equal triangle A B D is, for the same reason, half of the parallelogram B L therefore the square B G is equal to the parallelogram B L. In like manner the square H C may be shown to be equal to the parallelogram C L, and consequently the square C D is equal to the two squares B G, C H.

Ex.—Demonstrate that the two lines F C, A D, cut each other at right angles.

DEMONSTRATION II.



Draw B N at right angles to B C, meeting F G in N; draw C O at right angles to B C, meeting K H produced in O; join N O, and through A draw L A M, at right angles to B C. Join N A, O A.

1.—Then the triangle B F N is equal to B A C. (Why?) The triangle O K C is also equal to B A C. (Why?) Therefore the triangle B F N is equal to O K C, and B N and C O are equal to each other and to B C; hence B C O N is a square.

2.—The triangle A B N is half of the square B G, and also half of the rectangle B M L N; (why?) therefore the square B G is equal to the rectangle B M L N. In like manner the square C H may be shown to be equal to the rectangle L M C O by means of the triangle O A C. (The pupil ought to prove this.) Hence the two squares, B G and C H, are together equal to the square B O.

COR.—Hence, if the square described on one side of a triangle be equal to the squares described on the other two sides, the triangle is right-angled.

Ex.—Prove that the area of the triangle, D B F (fig. in Demon. I.) is equal to that of E C K.

EXERCISES.

1.—If the base of a right-angled triangle be 30 feet,

and the perpendicular 40, find, both by construction and calculation, the length of the hypothenuse.

2.—If the base of a right-angled triangle be 40 feet, and the hypothenuse 50, required by construction and calculation the length of the perpendicular.

3.—Required the side of a square which shall be equal to the sum of the squares described on three given lines.

4.—Required to find by construction the side of a square which shall be equal to the difference between the squares described on the lines A, C.

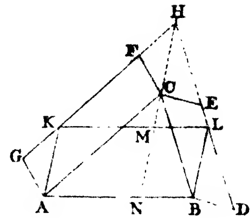
5.—Find by construction the side of a square whose area shall be five times the area of the square described on a given line.

6.—Find by construction the side of a square the area of which shall be half the area of the square described on a given line.

7.—The smallest whole numbers representing the sides of a right-angled triangle are 3, 4, 5. Employ this property in drawing a perpendicular to a given line, by using a scale of equal parts.

GENERAL PROPOSITION, EMBRACING THE 47TH AS A PARTICULAR CASE.

If any parallelograms be described on the two sides of any triangle, and if the sides of the parallelograms be produced to meet, and the point of intersection and the vertex of the triangle be joined, these parallelograms shall be equal to a parallelogram described on the base, and having two of its sides parallel to the line joining the point of intersection and the vertex, and limited by the sides of the two parallelograms.



Let A B C be the given triangle, and C D, C G, the parallelograms described on the sides. Produce D E, G F, to meet in H; join H C, and draw A K, B L, parallel to it, and join K L; then will the parallelogram A B L K be equal to C D and C G; for B L is obviously equal to C H, (why?) and A K is also equal to C H, therefore A K = B L, and the figure K B is a parallelogram.

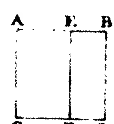
The parallelogram M N B L is equal to L B C H, which is equal to C B D E. Hence M N B L is equal to C B D E. In like manner A K M N may be shown to be equal to A G F C. (Why?) Hence the parallelogram A K L B is equal to G C and D C.

COR.—When the angle A C B becomes a right angle, and the parallelograms C D, C G squares, K B will also be a square. Hence the square described on the hypothenuse is equal to squares described on the other two sides.

SECTION XI.

ON THE EQUALITY OF SQUARES AND RECTANGLES CONSTRUCTED ON LINES AND THEIR SEGMENTS.

PROP. I.—If there be two lines, one of which is divided into any number of equal parts, the rectangle contained by these lines is equal to the sum of the rectangles contained by the undivided line and the several segments of the divided line.



Let A B, A C, be the lines, and let A B be divided into any number of parts, for example, in two parts, A E, E B, then the rectangle A D is *obviously* made up of the two rectangles A F, E D.

Ex.—If $AE=3$ and $EB=2$ and $AC=5$,

Then $5\times5=25$, the area of the rectangle AD.

And $5\times3=15$, the area of the rectangle AF.

And $5\times2=10$, the area of ED.

Hence $5\times5=5\times3+5\times2$.

Cor. 1.—If AB be equal to AC, then $AB\times AC$, or $AB\times AB$, or $AB^2=AB\times AE+AB\times EB$.

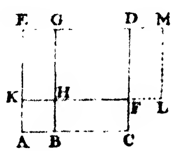
Note.—This property is expressed in common language as follows:—If a line be divided into any two parts, the square described on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

Cor. 2.—If the undivided line AC be equal to AE, one of the segments of the divided line, then

$$AB\times AC, \text{ or } AB\times AE=AE\times EB+AE^2.$$

Note.—The pupil is required to illustrate these two corollaries by numbers, and express the last property in words.

Prop. II.—The square described on the *sum* of two lines is equal to the squares of those lines, together with twice the rectangle contained by them.



Let AB, BC, be two lines, and AD the square described on their sum; draw BG parallel to CD, make AK equal to AB, and draw KF parallel to AC; then it is obvious that the square AD is divided into two squares AH, HD, being the squares described on AB and HF, which is equal to BC, together with the two rectangles EH, HC, which are obviously the rectangles contained by AB, BC.

This property may be expressed in general terms as follows:—

$$(AB+BC)^2=AB^2+2\,AB\times BC+BC^2.$$

Ex.—If $AB=3$ and BC 7, then $10^2=3^2+2\times3\times7+7^2$;

or $100=9+42+49$.

Cor. 1.—If $AB=BC$, it follows that the square described on a line is equal to *four* times the square described on half the line.

COR. 2.—If from the sum of the squares A D, H D, we take away the rectangle E F and the rectangle G C, there will remain the square A H, but A B is the difference between the two lines A C, B C, and the rectangles E F, G C, are the rectangles contained by A C, B C; hence the following property:—The square described on the difference between two lines is equal to the squares of those lines diminished by *twice* their rectangle,

Generally $(AC-BC)^2 = AC^2 - 2\,AC \times BC + BC^2$.

Numerically $(7-3)^2 = 7^2 - 2\times 7\times 3 + 3^2$.

or, $16 = 49 - 42 + 9$.

COR. 3.—The square A D diminished by the square A H leaves the rectangle E F, together with H C. If H C be placed in the position D L, it will form, with E F, the rectangle E L, which is obviously contained by K L, which is equal to the *sum* of A C and A B, and E K, which is their difference; hence the following important property:—The difference between the squares of two lines is equal to the rectangle contained by their *sum* and *difference*:

Generally $AC^2 - BC^2 = (AC + BC)\,(AC - BC)$.

Numerically $7^2 - 3^2 = (7+3)\,(7-3)$,

or $49 - 9 = 40 = 10 \times 4$.

COR. 4.—If a line, A B, be bisected in C, and cut unequally in D, then, since $CB^2 - CD^2 = (CB + CD)\,(CB - CD) = AD \times DB$, or $CB^2 = AD \times DB + CD^2$; it follows that if a line be bisected and divided unequally in any other point, the rectangle contained by the two unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

EX.—Illustrate this by any numbers.

COR. 5.—If A B be bisected in C, and produced to any point D′, then since $CD'^2 - CB^2 = (CD' + CB)\,(CD' - CB) = AD' \times BD'$, or $CD'^2 = AD' \times BD' + CB^2$, it follows — that if a straight line be bisected and produced to any point, the rectangle contained by the whole line produced, and the part produced, together with the square of the line between the points of

section, is equal to the square of the line made up of the half and the part produced.

Ex.—Illustrate this property by any numbers.

COR. 6.—Since $AD^2 = (AC + CD)^2 = AC^2 + 2\,AC \times CD + CD$
And since $BD^2 = (BC - CD)^2 = (AC - CD)^2 = AC^2 - 2\,AC \times CD + CD^2$

By addition $AD^2 + BD^2 \quad = \quad 2\,AC^2 + 2\,CD^2$

But AC is half the sum of two of the lines AD, DB and CD half their difference. Hence the sum of the squares of two lines is equal to *twice* the square of half their sum, and *twice* the square of half their difference.

EXAMPLE.—If $AD = 7$ and $DB = 3$
Then $7^2 + 3^2 = 2 \times 5^2 + 2 \times 2^2$
Or $49 + 9 = 50 + 8$.

EXERCISES.

1.—Find the difference between the squares of 25 and 28, without squaring the numbers.

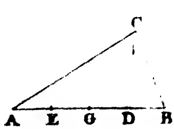
2.—The hypothenuse of a right-angled triangle is 150 feet, and the base 120; required the length of the perpendicular without *squaring* the given numbers.

APPLICATION OF SOME OF THE PRECEDING PROPERTIES TO THE FOLLOWING IMPORTANT PROBLEM.

PROB.—Given the three sides of a triangle to find the perpendicular.

CASE I.—When the perpendicular falls within the base.

From C let fall the perpendicular CD on AB;



then in the triangle ACD, $AC^2 = AD^2 + CD^2$,

and in the triangle BCD $BC^2 = BD^2 + CD^2$.

By subtraction $AC^2 - BC^2 = AD^2 - BD^2$.

Take $AE = BD$, and bisect AB in G. It is obvious that ED is also bisected in G, i.e. $2GD = ED$.

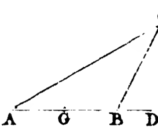
As the difference of the squares of two lines = the rectangle contained by their sum and difference,
$(AC+BC)\ (AC-BC) = (AD+DB)\ (AD-DB) = AB \times 2GD = 2AB \times GD$; or dividing both sides by $2AB$ we have

$$\overline{\quad\quad\quad 2AB.}$$

Hence AD and CD will be easily found.

Ex.—Let AB be 25, AC 20, and BC 15; bisect AB in G, then $GD = \frac{35 \times 5}{2 \times 25} = 3\frac{1}{2}$. Hence $AD = AG + GD = 12\frac{1}{2} + 3\frac{1}{2} = 16$. Consequently $CD = \sqrt{AC^2 - AD^2} = \sqrt{20^2 - 16^2} = 12$.

Case II.—When the perpendicular falls without the base.



$$\text{Since } AC^2 = AD^2 + CD^2,$$
$$\text{and } BC^2 = BD^2 + CD^2,$$
$$\text{By subtraction } \overline{AC^2} - \overline{BC^2} = \overline{AD^2} - \overline{BD^2},$$
$$\text{or, } (AC+BC)\ (AC-BC) = (AD+BD)\ (AD-BD).$$

Bisect AB in G, then $AD + BD = 2GD$, (why?) and $AD - BD$ is AB,

Hence $AB \times 2GD$ or $2AB \times GD = (AC+BC)\ (AC-BC)$

And dividing by $2AB$ we have $GD = \frac{(AC+BC)\ (AC-BC)}{2AB}$

Hence if from GD we take away GB, there remains BD, and consequently the length of CD is easily found.

Ex.—Let $AC = 20$, $CB = 16$, $AB = 8$, then $GD = \frac{36 \times 4}{16} = 9$; and $BD = 9 - 4 = 5$, and $CD = \sqrt{CB^2 - BD^2} = \sqrt{16^2 - 5^2} = \sqrt{231} = 15.19$.

As any angle of a triangle may be considered as the vertex, and the opposite side as the base, we may generalise the two cases thus:—If from the vertex of a

triangle a perpendicular be let fall on the base, or base produced, the distance from the foot of the perpendicular to the middle of the base is equal to the product of the sum and difference of the two sides of the triangle divided by twice the base.

Ex.—If the three sides of a triangle be respectively 100, 50, and 60 feet, required the lengths of the perpendiculars let fall from the three angular points on the opposite sides, and also the area of the triangle deduced by employing each of the perpendiculars.

SECTION XII.

ON THE MOST ESSENTIAL PROPERTIES OF THE SIMPLER SOLIDS BOUNDED BY PLANE SURFACES.

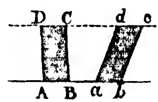
A *plane surface* is a figure bounded by *lines*. A *solid* is a figure bounded by surfaces. The simplest plane figure is a square, which is therefore taken as the measuring *unit* of surfaces. The simplest solid is a *cube*, or figure bounded by six equal squares, and is therefore chosen as the measuring *unit* of solids. When the length of one of the sides of the squares is one inch, the solid is called a cubic inch; when a foot, a cubic foot, &c.; and the whole of the mensuration of solids consists in finding how many cubic inches, cubic feet, &c. are contained in a solid of any form.

Def.—A *prism* is a solid bounded by two parallel and equal planes, the form of the other portion of the bounding surface depending on the figure of the parallel planes. Hence a cube is only a particular form of a prism. Any particular prism gets its name from

the figure of its parallel ends. If the ends are squares, it is called a *square* prism, or parallelopiped; if rectangles, a *rectangular prism*; if triangles, a *triangular prism*.

Obs.—Models of these solids, having the areas of the bases equal, and having equal altitudes, some of them standing perpendicular to the horizon, and others obliquely, should be shown to the pupil.

Prop. I.—All prisms standing on equal bases, and having the same altitude, or being between the same parallel planes, are equal to one another.



To prove this property clearly to a young pupil, discs of pasteboard of different shapes and equal surfaces should be procured. Some of them being squares, some rectangles, some triangles, and others circles. By building up these discs as in the annexed figure, the pupil will readily perceive that the same number must be employed in reaching the same height, whether they be built perpendicularly or obliquely. As each prism is composed of the same number of equal thin discs, whatever be its form, its *bulk*, *solidity*, *mass*, or *capacity*, must be the same in each. When the paper is thick, the *oblique* prisms will have their sides formed of ascending steps or stairs; but when the paper becomes indefinitely thin, these steps will become so small that the surface formed by them may be viewed as a plane surface.

Now, as the square prism will obviously contain as many cubes as there are square inches, square feet, &c. in its base, and lineal inches, feet, &c. in its height, the number will be found by multiplying the number of square inches, feet, &c. in its base, by the lineal inches, feet, &c. in its height. Hence the following general rule for all prisms.

Rule.—Multiply the area of the base by the perpendicular height.

Note.—In taking the dimensions of any figure, whether plane or solid, the measures should be given in feet, tenths and hundredths of a foot. A measuring rod or tape should be used, having the foot divided into ten equal parts, and each of those again into ten equal parts. The lengths of lines would then be expressed in whole numbers and decimal fractions, and the calculations reduced to the greatest degree of simplicity. The term *duodecimals* ought no longer to find a place in the mensuration of surfaces and solids.

EXERCISES.

1.—How many cubic inches are contained in a cubic foot?

2.—How many cubic feet in a cubic yard?

3.—How many cubic yards of earth will be dug out in forming a pond, the length being 120 yards, the breadth 60 yards, and the depth 2 yards?

4.—How many cubic feet are contained in a log of wood, the breadth of the end being 2 feet, the thickness 2 feet, and the length 15 feet.

5.—How many cubic feet and yards of earth will be dug out of a ditch 4 feet broad at the top, 1 foot at the bottom, and 5 feet deep, the length being 1200 feet?

Def.—A pyramid is a solid standing on a base of any form, and terminating in a point. It is named from the particular form of the base on which it stands. If the base be a square, it is called a *square* pyramid; if a triangle, a *triangular* pyramid, &c.

Prop. II.—Pyramids standing on equal bases, and having the same *altitude*, are equal to one another.

This may be proved to a young pupil by having a series of discs of paper, of various forms, but equal surfaces, gradually diminishing in size till they ter-

minate in a point. These being placed above each other, may be made to form *right* or *oblique* pyramids, which being composed of the same number of equal discs, will obviously be equal to each other.

Prop. III. If a triangular prism, made of wood, be cut by a saw from one of its angular points to the side of the triangle in the other end, and the largest portion be again cut in a similar manner, the prism will be divided into *three* pyramids, which will have equal bases and altitudes. Hence the solidity of a pyramid will be found by taking one-third of the solidity of the prism having the same base and altitude.

Note.—It would be exceedingly difficult to demonstrate this property to a young pupil by means of a figure drawn on paper. We would therefore recommend to every teacher of elementary geometry, to procure a large prism of wood cut into pyramids, and the property will be rendered remarkably simple and interesting.

EXERCISES.

1.—How many cubic feet are contained in a *square* pyramid, the side of the square being 25 feet, and *perpendicular* altitude 90 feet?

2.—How many solid feet are contained in a triangular pyramid, the base being an equilateral triangle, whose side is 30 feet, and the perpendicular height 50 feet?

SECTION XIII.

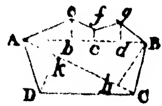
PRACTICAL APPLICATION OF THE PROPERTIES CONTAINED IN THE PRECEDING SECTIONS.

Land Surveying.—Land is measured by means of a *chain* sixty-six feet in length, divided into a hundred *links*. An acre contains ten square chains, or one hundred thousand square links. To give a very young pupil a clear idea of the magnitude of an acre, he may be told that if the length of *ten* chains be taken on a road whose breadth is *one* chain, that portion of the road will contain *exactly* an acre. In Scotland, land used to be measured by a chain about seventy-four feet and an inch long; but as that ought no longer to be employed, the *imperial* acre being used instead of the Scotch acre, we shall give examples in *imperial measures* as by law established. It would be better to express the area of a field in acres and decimal fractions of an acre, than by *acres, roods, poles, and yards.* But as these denominations, are still in use the pupil must recollect that $30\frac{1}{4}$ square yards make a *pole,* 40 poles a *rood,* and four roods an *acre.*

PROB. I.—To measure a *straight* line on the ground.

Place a pole at one end of the line to be measured, give the assistant who is to *lead* the chain, ten iron pins to be stuck in the ground successively at the end of each chain. The surveyor or measurer standing at one of the extremities of the line, directs the assistant, by waving his hand to either side, till he see the hand of the assistant, holding the chain, opposite the remote pole. The chain being stretched, and the iron pin pushed perpendicularly in the ground, the assistant goes on towards the remote pole, and the measurer having placed his eye directly above the iron pin, directs the assist-

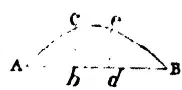
operation begins, and expressed in links. Hence the area being calculated in square links, divide by 100,000 for acres.



Ex.—Find the area of the annexed field, the number of links being as follows: A $b = 420$, A $c = 680$, A $d = 850$, A B 980, and the perpendicular $be = 360$, $cf = 250$, $dg = 280$. A C $= 1060$, B $h = 640$ D $k = 560$. These numbers are sufficient for calculating the area, but for constructing an accurate figure A h must be known, suppose 320.

Obs. 1.—The spaces cut off by the line A B are called *offsets.* It is sometimes necessary to measure a line in the adjacent field; the spaces between that line and the boundary of the field, are called *inlets.*

2.—When a portion of the boundary is a *curve*, as in the annexed figure, *small portions* may be viewed as nearly straight, and the area found accordingly.

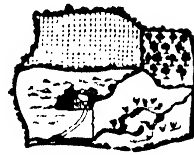


Ex.—What is the area of the field A c e B, A b being 320 links; A d 480; A B 690; $bc = 210$, and $de = 230$?

Obs. 3.—These examples will be sufficient to point out to the pupil the general mode of procedure; but without a little practice in the field, under the direction of an experienced teacher, a multitude of examples performed in the school-room will be of very little use.

4.—The pupil, having drawn a *correct* plan of the field actually measured, should be shown how to make a finished drawing of the plan, by representing in a natural manner *trees, hedges, rivers, lakes, roads, woods, bridges, houses, arable land, pasture land*, &c. The trees are shaded with green and a little yellow, the hedges green, pasture land green with a few slight touches of lines, woods green, roads reddish yellow, lakes and rivers slightly blue, &c.

The annexed figure will give the pupil an idea of the mode of procedure; but a single finished and coloured plan of an estate laid before him, will be more valuable than the most lengthened description.



2.—*Artificer's work.*—Under this term is generally introduced the practical modes of measuring the various parts of *houses;* as the *mason's*, the *bricklayer's*, the *carpenter's*, the *slater's*, *tiler's*, and the *glazier's work*, &c. As all this is merely the application of the simplest rules for plane surfaces and solids, the pupil requires only to be told the *conditions of agreement* to ascertain the sum which each artificer should receive for his work. In many schools where *Practical Mathematics* is taught, an immense portion of the pupil's valuable time is often consumed in the mere drudgery of calculating the sums which each artificer ought to receive for the different portions of a large building. A practice which cannot be too severely reprehended.

GENERAL OBSERVATIONS ON THE PRECEDING PART.

1.—Having in the preceding pages followed a plan which is considerably different from that *commonly* employed in *teaching* the Elements of Geometry, it may be necessary to give a few reasons for this mode of procedure, and at the same time to allude to the omission of several things, which, had they been introduced, would only have tended to perplex and retard the progress of the learner.

2.—The arrangement of the propositions which I have adopted, is that which appears to me best for the *purpose of instruction*, all minor considerations being made to give place to a quality which ought to be kept constantly in view in books intended for the instruction of youth. The properties are grouped together by obvious resemblances into sections, each of which may be viewed as constituting a distinct portion of elementary geometry. This division the author has found, from experience, to be well adapted for the purpose of instruction. Each section may be viewed as a separate lesson, which pupils of ordinary capacity will very soon master, and thus retain in the memory the assemblage of kindred properties as a whole.

3.—I have omitted the *formal* enunciation of self-evident truths, called *axioms*, being firmly convinced that there is no more necessity for stating them *formally* at the commencement of geometry, than at that of any other science. Some authors have gone so far as to consider a self-evident truth as a *proposition* requiring formal demonstration. This will appear to a pupil trifling with the subject. Place a sixpence, six penny-pieces, and twelve half-pence before him, and ask him whether the six penny-pieces or the twelve half-pence have the greatest value. He will at first suppose that you are not serious in asking him such a silly question. Having assured him that you are perfectly serious, you will still find it impossible to convince him that it requires any process of reasoning to prove that the six penny-pieces are equal in value to the twelve halfpence, since each of them is equal to the *sixpence*.

As axioms are still retained as an essential part of geometry by some authors, it may be necessary simply to mention them, that the pupil may know *verbally* what he already knows in reality.

AXIOMS.

1.—Magnitudes which are equal to the same, are equal to each other.

2.—If equals be added to equals, the sums will be equal.

3.—If equals be taken away from equals, the remainders will be equal.

4.—If equals be added to unequals, the sums will be unequal.

5.—If equals be taken away from unequals, the remainders will be unequal.

6.—The doubles of the same or equal magnitudes, are equal.

7.—The halves of the same or equal magnitudes, are equal.

8.—Magnitudes which coincide with one another, or exactly fill the same space, are equal.

9.—The whole is greater than its part.

10.—Two right lines cannot include a space.

11.—All right angles are equal.

4.—We have taken no notice of what are called *postulates* or *demands*, because common sense will show the pupil in each case what he may safely admit or grant, and what not. For example, if you ask him whether the direction of a straight line may be conceived to be continued as far as you please, he will answer in the affirmative. If you ask him whether he will grant that a large circle may be placed

within a smaller one, he will of course answer in the negative.

5.—I have in general avoided giving any demonstration of the *converse* of a proposition, because I was unwilling to break the chain of primary important properties, which naturally lead the pupil on to results of the highest interest. Another reason why I have avoided a formal demonstration of the converse of a proportion is, that the proof is most frequently of that kind what is called *indirect*, or a *reductio ad absurdum*, that is, showing it would be absurd to suppose the contrary. Pupils have a dislike to this kind of demonstration, and for this as well as other reasons it should be used very sparingly in geometry. Besides, in every case, the *real* converse of a proposition is necessarily true, the demonstration of the proposition establishing the truth of the converse.

Thus, for example, if it be proved that the equality of two of the angles of a triangle depends *essentially* on the *equality* of the opposite sides, it follows that the equality of the sides depends *essentially* on the equality of the angles.

If the pupil clearly understand the reasoning in the demonstration of a proposition, he will at once see whether what appears to be the converse is necessarily true or not. If, for example, the proposition were the following: If one angle of a triangle be obtuse, the other two angles are acute.

The converse of this proposition would seem to be the following. If a triangle have two of its angles acute, the remaining angle is obtuse. Now, no pupil who attends to the proof of the first proposition will admit the truth of this conclusion; for he will perceive that the conclusion in the first results *essentially*

from the property that the sum of the three angles is equal to two right angles. It does not therefore follow that the third angle of a triangle is obtuse, unless the sum of the two acute angles be less than a right angle. Besides, it will be an excellent discipline for the reasoning powers of the young pupil to determine the *real converse* of a proposition which must in every case be necessarily true.

EXERCISES.

1.—The *surfaces* of two triangles are equal when the three sides of the one are respectively equal to those of the other. When the *surfaces* of two triangles are equal, are the three sides of the one *necessarily* equal to the three sides of the other?

2.—If the greatest side of every triangle lie opposite the greater angle, does the greatest angle *necessarily* lie opposite the greatest side?

3.—If the square described on the hypothenuse of a *right* angled triangle be equal to the sum of the squares described on the other two sides, does it follow that when the sum of the squares described on two sides of a triangle is equal to the square described on the third side, the angle opposite the longest side must *necessarily* be a right angle?

In this case the pupil must observe from the demonstration of the first proposition, whether the conclusion *necessarily* depends on the circumstance, that one of the angles of the triangle is a right angle, *and on that circumstance alone.*

Though the direct mode of demonstration is to be preferred, yet in particular cases the indirect is sometimes absolutely necessary. It will therefore be use-

ful, before advancing farther, to give the pupil a clear view of this mode of reasoning. This will be best done by a few simple examples.

EXAMPLES.

1.—If it were proved, in a court of justice, that a certain crime could only have been committed by one or more of three persons, A, B, C, and if further evidence proved that neither A nor B could have committed it, it follows that C was the guilty person. This is a case of indirect reasoning.

2.—If there be two quantities A and B, and if when A is assumed to be greater than B, the supposition leads to an absurdity or impossibility; again, when A is assumed to be equal to B, and a process of accurate reasoning, founded on the supposition, also leads to an equally absurd or impossible result, it necessarily follows that A is less than B.

3.—What is the greatest number which will divide 28 and 36 without a remainder?

```
28)36(1
   28
   --
    8)28(3
      24
      --
       4)8(2
         8
         -
         0
```

According to the rule laid down in books of Arithmetic, 4 is the greatest common measure. Now in order to prove that 4 is the greatest common measure, we may proceed *indirectly* as follows: Suppose that 4 is not the greatest common measure of 28 and 36.

Then since the number which is supposed to be greater than 4 divides 28 and 36 without leaving a remainder, it will divide their difference or 8. Again, since it divides 8, it will divide 3 times 8, or 24, and also the difference of 28 and 24, or 4. But it is impossible that a number greater than 4 can divide 4 without a remainder, therefore 4 is the greatest common measure.

4.—If a triangle have two equal angles, it will have two equal sides.

Let $\angle$ B A D $=$ $\angle$ A B D (fig. 1st page 19), then if A D be not equal to B D, it must be either greater or less. Suppose it greater, and B C equal to A D. Then in the two triangles B A D, B A C, we have A B, A D in the first equal to A B, B C in the second, and the angle B A D in the first equal to A B C in the second, therefore the triangle B A D is equal to the triangle B A C, the less to the greater, which is absurd. In like manner, it may be shown that it will lead to an equally absurd conclusion to suppose that A D is less than B D. Hence, A D must be equal to B D.

Exercises requiring the indirect mode of demonstration, which the pupil may demonstrate if the teacher think it necessary.

I.—If a triangle have two unequal angles, it is required to prove that it has two unequal sides, and that the greater side lies opposite the greater angles.

II.—If there be two parallelograms, equal to each other in surface, and which stand on equal bases, it is required to prove that they have equal altitude, or are between the same parallels.

5.—But one of the greatest innovations which I have ventured to introduce, is the attempt to accustom pupils to reason by *Analysis* from the establishment of

the very first principles of the science. If the solution of geometrical exercises be universally acknowledged to be one of the finest modes of developing the reasoning powers of youth, it appears to me that the pupil can not be too soon made acquainted with the nature and use of the instrument which he is to employ in his investigations. It appears to me an act of cruelty on the part of a teacher, to give a pupil, a series of geometrical problems to solve, and theorems to investigate, without first giving him the means which are absolutely necessary to enable him to arrive at the results.

It is the same kind of cruelty as in a *taskmaster* forcing his bondsmen to make "bricks without giving them straw."

In reading the volume for the first time, it is not necessary to make the pupil solve all the exercises which have been given. This must necessarily be left to be determined by the age and abilities of the pupil.

PART II.

ON THE PROPORTION OF NUMBERS, LINES, AND SURFACES.

SECTION I.

ON THE FUNDAMENTAL PROPERTIES OF PROPORTIONAL NUMBERS.

1.—If there be any two numbers, 3 and 5 for example, then 3 is said to have the same proportion to 5 as the double of 3 to the double of 5, or as any number of times the first to the same number of times the second. This relation is expressed as follows: 3 is to 5, as 6 is to 10.

Since $\frac{3}{5}=\frac{6}{10}$ or, by using the sign of division, $3 \div 5 = 6 \div 10$, we have four numbers in proportion when the first divided by the second is equal to the third divided by the fourth. If we take away the line between the two points in the sign of division, it becomes $:$, and if we retain merely the extremities of the lines in the sign of equality it becomes $::$, hence $3 \div 5 = 6 \div 10$ may be written $3 : 5 :: 6 : 10$. This is the origin of the proportional points.

2.—If four numbers be proportional, the first is said to have the same *ratio* to the second that the third has to the fourth. Of proportional numbers, the first is called an *antecedent* and the second its *consequent*, the third an *antecedent* and the fourth its *consequent*. The first and fourth terms are called the *extremes*, the second and third the *means*.

When the second and third terms are the same, the last term is called a *third* proportional to the first and second, and the second a *mean proportional* between the first and last.

Thus, if 4 : 6 : : 6 : 9, then, 9 is a third proportional to 4 and 6, and 6 is a mean proportional between 4 and 9.

Prop. I.—If four numbers be proportional, the product of the extremes is equal to the product of the means.

If 3 : 5 : : 6 : 10. Then $3 \times 10 = 6 \times 5$.

For since $\frac{3}{5} = \frac{6}{10}$, by reducing them to a common denominator $\frac{3 \times 10}{5 \times 10} = \frac{6 \times 5}{10 \times 5}$, and taking away the common divisor or denominator, we have $3 \times 10 = 6 \times 5$.

Cor.—If 3 : 5 : : 6 : 10
Then 3 : 5 : $\frac{1}{10}$: $\frac{1}{6}$ (why?)

Def.—The numbers 10 and 6 are then said to be reciprocally proportional to 3 and 5.

EXERCISES.

1.—Given the first, second, and third terms, required a *rule* for finding the fourth? Given the first, third, and fourth, required the second? Given the first, second, and fourth, required the third? Given the second, third, and fourth, required the first?

2.—What is the third proportional to 2 and 4?

3.—Required a mean proportional between 2 and 8?

4.—Required the *rule* for finding a mean proportional between two given numbers.

5.—If there be two equal products composed of two factors each, it is required to place the terms in the

order of proportionals. Thus, if $6 \times 8 = 4 \times 12$, it is required to arrange the numbers proportionally, beginning with any one of them.

PROP. II.—If the corresponding terms of two or more sets of proportional numbers be multiplied together, the products will be proportional.

Thus, if	3 :	5 ::	6 :	10
And	2 :	4 ::	8 :	16
Then	6 :	20 ::	48 :	160

For $\frac{3}{5} = \frac{6}{10}$ and $\frac{2}{4} = \frac{8}{16}$. Hence $\frac{3}{5} \times \frac{2}{4} = \frac{6}{10} \times \frac{8}{16}$. That is $\frac{6}{20} = \frac{48}{160}$, or 6 : 20 :: 48 : 160.

COR. 1.—Hence, if there be two sets of proportional numbers, the quotient of the corresponding terms are proportional.

Thus, if	10 :	15 ::	40 :	60
And	2 :	5 ::	4 :	10
Then	5 :	3 ::	10 :	6

2.—Hence, if the first and second terms of a proportion be multiplied or divided by any two numbers, and the third and fourth by the same numbers, the products or quotients will be proportional.

For if	6 :	12 ::	8 :	16
And	2 :	3 ::	2 :	3
Then	2×6 :	12×3 ::	8×2 :	$16 + 3$
Or	[illegible] :	[illegible] ::	[illegible] :	[illegible].

3.—Hence, if the first and second, or third and fourth terms have a common multiplier or divisor, it may be taken away from each.

4.—Hence, if four numbers be proportional, their squares, cubes, &c. are proportional.

Thus, if	3 :	5 ::	6 :	10
Then	9 :	25 ::	36 :	100, &c.

5.—Hence also, their square or cube roots, &c. are proportional.

Thus, if $9 : 25 :: 36 : 100$

Then $\sqrt{9} : \sqrt{25} :: \sqrt{36} : \sqrt{100}$

That is $3 : 5 :: 6 : 10.$

Prop. III.—If four numbers be proportional, the first will be to the sum of the first and second, as the third to the sum of the third and fourth.

For if $3 : 5 :: 6 : 10$, then $3 \times 10 = 5 \times 6$, then by adding 3×6 to both products, we have $3 \times 10 + 3 \times 6 = 5 \times 6 + 3 \times 6$, or since 3 is a common multiplier in the first and second products, and 6 a common multiplier in the third and fourth, they may be written thus $3\,(10+6) = 6\,(5+3)$

That is $3 \times 16 = 6 \times 8.$

Hence $3 : 8 :: 6 : 16.$

Cor.—By taking the difference of the products 3×10, 3×6 and 5×6, 3×6, we have the following property. The first is to the difference of the first and second as the third is to the difference of the third and fourth.

Gen. Obs.—Whatever expression with regard to sum or difference, we apply to an antecedent and its consequent, if the same expression be applied to the other antecedent and its consequent, we shall have proportional numbers. These are arranged as follows, and may be easily proved as in the last proposition.

If $5 : 3 :: 10 : 6$

Then $5 : 5 + 3 :: 10 : 10 + 6$

$5 : 5 - 3 :: 10 : 10 - 6$

$5+3 : 5 - 3 :: 10 + 6 : 10 - 6.$

EXERCISES.

1.—Divide 20 into two parts, which shall be to each other as 2 to 3.

2.—Divide 100 into three parts, which shall be to one another as 2, 3, 5.

Prop. IV.—If there be a series of numbers in continued proportion, then one of the antecedents will be to its consequent as the sum of all the antecedents to the sum of all the consequents.

Thus, if $2 : 4 :: 3 : 6 :: 5 : 10$
Then $2 : 4 :: 2 + 3 + 5 : 4 + 6 + 10$.

This must obviously be the case, for each of the numbers 4, 6, 10 in the fourth being to each of those in the third terms namely, 2, 3, 5 in the proportion of 2 to 4, their sum must be in the same proportion.

Prop. V.—If there be *three* numbers in continued proportion, then the first will be to the third as the *square* of the first to the *square* of the second.

For if $3 : 6 :: 6 : 12$
$6^2 = 3 \times 12$ and multiplying both by 3
We have $3 \times 6^2 = 3^2 \times 12$
Hence $3 : 12 :: 3^2 : 6^2$

Cor.—If there be *four* numbers in continued proportion, the first will be to the fourth as the *cube* of the first to the *cube* of the second.

For let 3, 6, 12, 24 be the numbers.
Then by the Prop. $3 : 12 :: 3^2 : 6^2$
But $12 : 24 :: 3 : 6$

By mult. $3 \times 12 : 24 \times 12 :: 3^3 : 6^3$
Or $3 : 24 :: 3^3 : 6^3$

In like manner it may be shown of five numbers, 3, 6, 12, 24, 48 in continued proportion, that $3 : 48 :: 3^4 : 6^4$, &c.

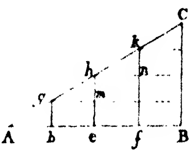
General Remark.—Though the preceding properties have been proved by using numbers, yet these numbers being employed merely as the representatives of any quantities whatever as lines, surfaces, &c. the properties obviously belong to all quantities, whether they be represented by numbers or not.

SECTION II.

ON THE FUNDAMENTAL PROPERTIES OF PROPORTIONAL LINES.

PROP. I.—The segments into which diverging lines are cut by parallels are proportional to one another.

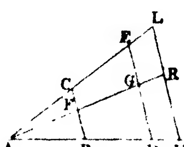
Let A C be divided into any number of equal parts, and C B joined, and the lines kf, he, &c. drawn parallel to B C, then A B will be divided into the same number of equal parts.



Hence A b : A g : : A e : A h : : A f : A k, : : A B : A C.

NOTE.—Though the above demonstration appear to apply only to those cases in which the line A g is contained an exact numbers of times in A h or A k or A C, it may easily be generalised by supposing A g divided into an indefinitely great number of parts, so that each of these indefinitely small portions may be contained exactly a certain number of times in A h, or A k, or A C.

Hence generally A B : A C : : A D : A E : : A H : A L, &c.



COR. 1.—Since A B : A D : : A C : A E, we have A B : A D — A B : : A C : A E — A C. That is A B : B D : : A C : C E : : D H : E L.

COR. 2.—If lines be drawn through the points of division g, h, k, &c. parallel to A B, it is obvious that h e will be the double of g b, k f the triple of g b, &c., and as this holds whatever be the number of equal parts, we have the following general property.

Parallel lines are in the same proportion as the segments into which they divide diverging lines. Thus in the last figure, A C : A E : : C B : E D.

EXERCISES.

1.—The pupil is required to write down all the proportions which exist between the segments of the lines in the last figure.

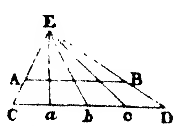
2.—Find by construction a line, A E, which shall be a fourth, proportional to three given lines, A B, A C, A D. (See last fig.)

3.—Find by construction a third proportional to any two given lines.

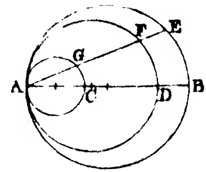
4.—Find by means of the diagonal scale a fourth proportional to three given numbers, 248, 372, and 320.

5.—If 24 yards of cloth cost £32, find by means of the diagonal scale the price of 18 yards.

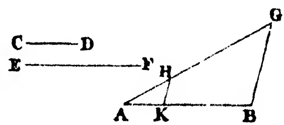
6.—To divide a line, A B, into any number of equal parts: draw a line, C D, parallel to it; take a distance, *c a*, and repeat it along C D the number of times required, *c a* being taken of such a length that C D shall be considerably longer than A B; join C A, and produce it to meet D B produced in E; join E *a*, E *b*, E *c*, and these lines will divide A B into the required number of equal parts. Required the demonstration.



7.—If A B, the diameter of a circle, be divided into any number of parts, A C, A D, and if on A C, A D, as diameters other circles be described, it is required to prove that any chord, A E, drawn from the point A, is cut by the circumferences of the circles in the same ratio as the segments of A B; that is, A C : A D :: AG : A F, &c.



8.—Divide a line, A B, into two parts, which shall be to each other as two lines, C D, E F.



Draw any diverging line, A G; make A H = C D and H G = E F, join G B, and draw H K parallel to it; then A K will be to K B as A H to H G.

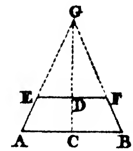
9.—Divide a given line into four parts, which shall be to each other as the natural numbers 1, 2, 3, 4.

10.—It is required to find a number, such that if 12 be added to it, the sum shall be to the number required as 3 is to 2.

Since 3 : 2 :: 12 + the required number : req[d] numb. we have 3—2 or 1 : 2 :: 12 : 24 the number required.

11.—It is required to produce a given line, so that the whole line produced shall be to the part produced in a given ratio.

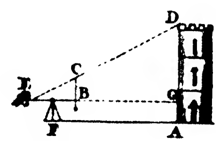
12.—Given the base A B of an isosceles triangle, the length of the line C D drawn from C, the middle of the base at right angles to A B, and the length of E F parallel to the base; it is required to find both by construction and calculation, the length of the whole perpendicular C G of the triangle.



PRACTICAL APPLICATIONS OF THE PRECEDING PROPERTIES.

PROB. I.—To find the height of a perpendicular object by means of two rods placed at right angles to one another.

Let E B, B C be two straight rods forming a right angle at B. Place it on a stick or three feet F, so that the eye may be conveniently applied at E. Place C B perpendicular by means of a plumb line hung from C. If it be so placed that the top of the object D be seen in the same straight line with E and C, and if the distance E G be measured, we have E B : B C :: E G : G D, which being increased by the height of the eye, that is, by A G, will give the height of the object.



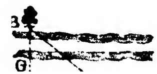
EXERCISES.

1.—If E B be 25 inches, and B C 25 inches, and E G 40 feet, required the height of the tower, the height of E B above the horizontal ground being 4 feet?

2.—If B C be 30 inches, the other numbers remaining the same, required the height of the tower?

PROB. II.—To find the breadth of a river without crossing it.

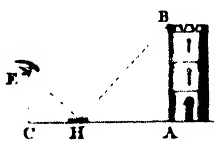
From a point A near the brink, observe an object B on the opposite bank, place a pole at C by means of the cross staff or theodolite, so that the angle B A C may be a right angle. Place another pole at D, so that C D may be at right angles to A C. Standing at D, make an assistant plant a pole at E in the line A C, and in the direction D B. The distances A E, E C, C D, being measured, and found to be, for example, A E 120 feet, E C 180, and D C 140, required the breadth of the river G B, A G being 12 feet?



PROB. III.—To find the height of a perpendicular object by means of an *artificial horizon*.

An artificial horizon is a horizontal surface of any substance capable of reflecting light uniformly, as the surface of mercury, ink, &c. If soft treacle be poured into a saucer, it will form a very good artificial horizon.

Place the artificial horizon at H, and the eye at E, so that the top of the object may be seen by reflection in H. Measure the height C E of the eye, and the distances C H, H A, then C H : C E : : A H : A B.



PROB. IV.—To find the height of a perpendicular object by means of its shadow.

Place a stick perpendicularly in level ground and measure the length of its shadow, measure also the length of the shadow of the object, and you will obviously have the following proportion:

As the shadow of the stick : height of the stick : : shadow of the object : height of the object. (Why?)

Thales is said to have taught the Egyptians how to measure the height of the pyramids by means of their shadows.

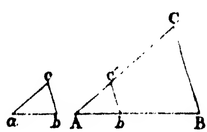
SECTION III.

ON THE PROPORTIONS OF THE CORRESPONDING SIDES AND THE PERIMETERS OF SIMILAR FIGURES.

DEF.—Two triangles are said to be *similar* to one another when all the angles of the one are respectively equal to those of the other. Those sides which are opposite equal angles are called homologous sides. We shall adopt the simpler term *corresponding* sides.

PROP. I.—The corresponding sides of similar triangles are proportional to each other.

Let $a\,b\,c$, $A\,B\,C$, be similar triangles; place $a\,b\,c$ in the position $A\,b'c'$, then $c'\,b'$ is obviously parallel to $C\,B$, (why?)



Hence $A\,b' : b'\,c' :: A\,B : B\,C$

And $A\,b' : A\,c' :: A\,B : A\,C$

Also $A\,c' : b'\,c' :: A\,C : B\,C$

COR. 1.—Since the *proportionality* of the sides of two triangles depends essentially on the *equality* of their angles, it follows that the equality of the angles depends essentially on the proportion of the sides. Hence two triangles are similar when the three sides of the one are proportional to the three sides of the other.

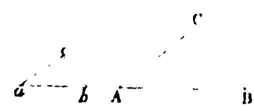
COR. 2.—When $A\,c'$ is to $A\,b'$, as $A\,C$ to $A\,B$, it necessarily follows that $b'\,c'$ is parallel to $B\,C$. Hence two triangles $a\,b\,c$, $A\,B\,C$ are similar when they have an angle in the one equal to an angle in the other, and the sides containing the equal angles proportional.

COR. 3.—When angle a is equal to angle A, and when $a\,c$ is to $c\,b$ as $A\,C$ to $C\,B$, it does not necessarily follow that the triangle $a\,b\,c$ is similar to $A\,B\,C$. For if $a\,b\,c$ be placed in the position $A\,b'\,c'$, then $b'\,c'$ may be parallel to $C\,B$, in which case

the angle b' is equal to the angle B, and the triangles are similar. But $c'b'$ may have another position nearer A, in which case it is not parallel to CB, and consequently the triangles are not similar. Again, CB may also have a different position, in which case it will be parallel to the new position of $c'b'$, the angles b', B being then equal and obtuse. Hence, when two triangles have two sides of the one proportional to two sides of the other, and an angle *opposite* one of these sides in the one, equal to the angle opposite the corresponding side in the other, these triangles will be similar when they are both acute angled, both right angled, or both obtuse angled.

NOTE.—The pupil will here observe that if he change the term *equal* to *proportional* the conditions which render triangles equal, (see section II. Part I.) are the same as those which make them similar.

PROP. II.—In similar triangles, the *perimeters* or sums of the sides are proportional to the corresponding or homologous sides.



Let abc, ABC be similar triangles.

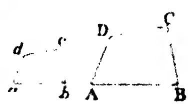
Then $ac : AC :: ab : AB :: cb : CB$

Hence $ac : AC :: ac + ab + cb : AC + AB + CB$.

COR.—Hence if similar figures be similarly placed so that a side of the one is parallel to a corresponding side of the other, the remaining sides of the one are parallel to those of the other.

PROP. III.—In similar rectilineal figures, the *perimeters* are as the corresponding sides.

DEF.—Similar rectilineal figures are made up of the same number of similar triangles similarly placed.



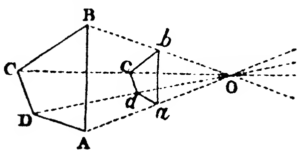
Let $abcd$, $ABCD$, be similar figures.

Then $ab : AB :: bc : BC :: cd : CD : ad : AD$
Hence $ab : AB :: ab + bc + cd + ad : AB + BC + CD + AD$.

Cor. 1.—$bd : BD ::$ perimeter of $abcd$: perimeter of $ABCD$.

Cor. 2.—Since circles may be viewed as regular polygons, containing an indefinitely great number of sides, it follows that the circumferences of circles are as their diameters.

Prop. IV.—If two similar figures be placed with the sides of the one parallel to the sides of the other, and if the corresponding angular points be joined and produced, the lines will all meet in the same point.



Let $ABCD$, $abcd$ be two similar figures having their sides parallel. Join the corresponding angular points and produce the lines, they will all meet in the same point O.

For since $CB : cb :: CD : cd$

And since $CB : cb :: BO : bo :: CO : co :: DO : do$, it follows that the three lines Bb, Cc, Dd, being produced, will all meet in the same point. In like manner it may be shown that all the other lines joining the corresponding angular points, being produced will meet in the same point.

Note.—If the figure $ABCD$ be placed on the right of O, in a reversed and inverted position, at the same distance, and with its sides parallel to those of $abcd$, it may be easily shown that the same lines would still meet in one point.

Def.—This remarkable point O is called the *centre of similitude* or *similarity*.

Ex.—Given any figure $ABCD$, it is required to describe a similar figure $abcd$, by means of this property, so that each of its sides shall be half of those in the original, or in any given proportion.

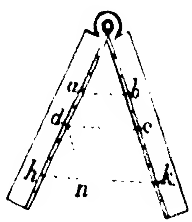
Ex. 2.—Given any rectilineal figure; it is required to describe another similar figure on a given line. The construction to be founded on a different property from the last.

PRACTICAL APPLICATION OF THE PRECEDING PROPERTIES.

I.—Description and use of the sector.

The sector consists of two equal branches moveable about a centre, and divided into any number of equal parts, beginning at the centre.

PROB. I.—To find a fourth proportional to three given lines, which we may call A, B, C.



Take O a, O b each equal to A, and open the sector till $a\,b$ be equal to B; take O d, O c each equal to C, then the distance $d\,c$ will be the fourth proportional required.

For O a : $a\,b$ :: O d : $d\,c$.

Ex.—Find a fourth proportional to 3, 4, 5.

PROB. II.—To divide a given line, A, into two parts, which shall be to each other as two numbers or lines, suppose 3 to 5.

Take O h, O k each equal to the sum of 3 and 5, or 8, open the sector till $h\,k$ be equal to the given line A: then the distance $a\,b$ between the points 3, 3, will be one of the parts, and $d\,c$, the distance between the points 5, 5, will be the other part.

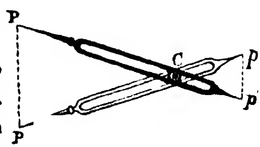
For draw $d\,n$ parallel to O k, then the triangle O $a\,b$ will be equal to $d\,h\,n$, (why?) and $d\,c = n\,k$; but $a\,b$ is to $d\,c$ as 3 to 5, therefore $h\,n$ is to $n\,k$ as 3 to 5.

EXERCISES.

1.—Divide 70 into two parts which shall be to one another as 2 to 3.

2.—Divide 100 into three parts which shall be to each other as 2, 3, 5.

II.—Description and use of the *proportional compasses*.



The proportional compasses consists of two equal branches or legs P′p′, P p divided into equal parts. The centre C, about which the legs turn, may be altered at pleasure so as to give any proportion between the parts P C, C *p*.

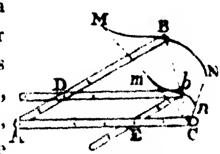
USE.—One of the principal uses of the proportional compasses is to reduce or enlarge the plan of a field, estate, &c. so that the sides may have any assigned proportion to each other.

Let A B C D be a figure (see fig. page 101) it is required to reduce it so that the sides of the new figure shall be to those of the original as any two numbers *m*, *n*. Alter the centre so that *p* C shall be to P C as *m* to *n*. Take the distance A B between the points P P′, then the distance *pp*′ will give the side *a b*, (why ?) Do the same with A C, B C, and construct the triangle *a b c* with the distances thus found between *pp*′, and the triangle *a b c* shall be similar to A B C. Do the same with the triangle A C D, and make *a c d* similar to it, and so on till the plan be completed.

NOTE.—By the application of this property, *profiles* may be taken. The point P of the larger branch of a rod moveable

about a universal centre, is moved over the forehead, nose, &c. whilst the shorter end, having a pencil fixed in it and pressing against a card, will obviously trace out a similar outline.

III.—Description and use of the *pantagraph.*



The pantagraph consists of a parallelogram A D *b* E of wood or metal moveable about the points A, E, *b*, D. The branches A B, A C, of which A D, A E are portions, are equal to each other, and the side D *b* is equal to D B and E C to E *b*. A sharp point is fixed at C, which rests on a drawing board, or on a piece of lead placed on the board. A sharp tracing point is fixed at B, and a pencil or drawing point at *b*. It is obvious that the points B, *b*, C will always be in a straight line, whether the angle A be increased or diminished, and that the tracing branch A B will in every position be parallel to the drawing branch E *b*. Hence if the tracing point B be made to pass over a straight line, the drawing point *b* will also pass over a straight line parallel to it, and in the proportion of E C to A C. Hence, if any plan or line M B N be placed in the position marked in the figure, and the tracing point made to pass over the outline of the figure, the drawing point will trace out a similar figure.

If the tracing point be fixed at *b* and the drawing point at B, the original plan may be enlarged in any proportion.

If the sharp point be fixed at *b*, the tracing point at B, and the drawing point at C, then if A E be made equal to E C, the plan being placed at B, may be accurately copied by the drawing point C. In

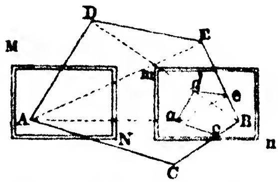
the first and second case, c is the centre of similitude. In the last *b* becomes that centre. Instead of the point being attached to the instrument, a vertical wire may be fixed in a flat piece of lead, and made to pass through one of the holes in one of the branches.

NOTE.—Professor Wallace, of Edinburgh, has invented an instrument, called an *Eidograph*, in which two arms are made to turn round, keeping always parallel to each, and being armed with tracing and drawing points, reduces, enlarges, or copies plans in the same way and on the same principles as the Pantagraph.

IV.—Description and use of the *plain table.*

The plain table consists of a rectangular piece of wood of any convenient size, suppose a foot broad and fifteen inches long. The paper on which the plan is to be drawn, may be damped and pasted round the edges of the plain table. A flat ruler, having two sights fixed perpendicularly on it, one of them having a small hole about the size of a small pin, and the other having a large hole with two cross hairs or wires fixed at right angles to one another, is used for drawing the direction of lines to remarkable points. The table is made to rest horizontally on three feet in the usual way. The instrument is sometimes fitted more expensively by having a moveable frame round the board, to hold the paper tight, and divided into equal parts. A compass needle is sometimes added.

This instrument may be used in taking the plan of a field, bounded by straight lines, by placing the instrument at two remarkable points, and drawing lines as in the annexed figure.



Let A C B E D be a field, the plan of which is to be taken. Place the plain table in the position M N at one of the angles A, and having fixed a pin perpendicularly in the table at A, or made a point on the table, direct the sights to B, and draw the line A B; direct the sights in succession to C, E, D, and draw the lines on the table in the directions of those points. Remove the table to B, and place it horizontally. Take a point B, and place the table so that the same side of the ruler being applied to the line B A, the pole placed at A may be seen through the sights. Turn the ruler in succession about the point B till the poles at C, E, D be seen through the sights, and draw the lines B *c*, B *e*, B *d* in the several directions, and these lines intersecting the former in the points *c*, *e*, *d*, will determine the angular points of the plan. The points *e d*, *d a*, and *a c*, being joined, will obviously form a figure *a c* B *e d* similar to A C B E D, (why?)

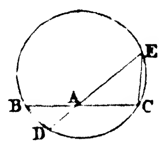
NOTE.—If the length of the side A B be measured by means of a chain, and *a* B be taken from the diagonal scale, having the same number of equal parts as A B has links, the lengths of the other sides may be found from the same scale, and the area of the field may be computed.

We would advise the young surveyor not to employ this method in finding the *area* of a field, as it is apt to lead to very serious errors without the possibility of detecting them.

SECTION IV.

ON THE PROPORTIONS OF CERTAIN LINES CUTTING AND TOUCHING A CIRCLE.

PROP. I.—If two lines cut each other within a circle, it is required to find the proportion which exists between the segments.

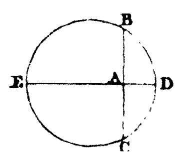


Let B C, D E be the lines; join B D, C E. Then the triangles B A D, E A C are obviously similar, (why?) Therefore

A B : A D : : A E : A C.

COR. 1.—Since A B × A C = A D × A E, it follows that a rectangle having one of its sides equal to A B, and the other equal to A C, will be equal to a rectangle having A D for one of its sides, and A E for its adjacent side.

EX.—If the line A B be 4 inches, A C 12, and A E 16, required the length of A D, so that a circle may pass through the four points B D C E?



COR. 2.—If one of the lines D E pass through the centre and cut any chord B C at right angles, then

A E × A D = A B × A C = A B^2.

That is, the rectangle having A E, A D for its adjacent sides, is equal to the square described on A B.

EXERCISES.

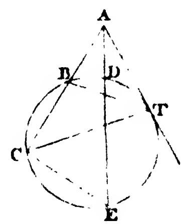
1.—Find by construction and calculation the side of a square which shall be equal to a rectangle, one of the sides, E A, being 9 feet, and the other, A D, 4?

2.—Given the side of a square A B (see last fig.)

and the length of one side of rectangle A D, it is required to find (by means of the property that the angle in a semicircle is a right angle) the length of the other side of the rectangle A E, so that its surface shall be equal to that of the square?

3.—Given the length of B C, the chord of an arc 10 feet, and A D the perpendicular from the middle of the chord to the arc, 4 feet, required by calculation the diameter of the circle of which it is an arc.

PROP. II.—It is required to find the relation of lines which cut each other without a circle.



Let A C, A E be the lines. Join B D, (the pupil will do so, as the line as been omitted), and C E, then B D E C being a four-sided figure inscribed in a circle, the opposite angles at D and C are together equal to two right angles; but the two, A D B, B D E, are also equal to two right angles, consequently the angle A C E is equal to the angle A D B. Hence all the angles of the triangle A B D are respectively equal to those of A E C, (why?) and therefore A C : A E : : A D : A B.

COR. 1.—Since A B $\times$ A C $=$ A D $\times$ A E, a rectangle having A B, A C for its sides, will be equal to a rectangle having A D, A E on its sides.

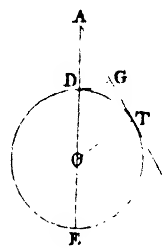
EX.—If A B be 4 inches, B C 5, and A D 3, it is required to find the length of D E, so that a circle may pass through the four points B, C, E, D?

COR. 2.—If one of the lines A E be supposed to turn round the point A towards the right, the chord D E will constantly diminish till it vanish altogether. The two lines A D, A E will become more and more nearly of the same length, and consequently the rectangle contained by A D, A E will approach to a square, which it will become when the chord D E vanishes, that is, when A T becomes a tangent to the circle.

Hence in that case $AB \times AC = AT^2$, that is, *the rectangle contained by the whole line which cuts the circle, and the part without the circle is equal to the square of the tangent.*

NOTE.—As this property is of great importance, we shall prove it in a different manner.

Join BT and CT. Since AT touches the circle and TB cuts it, the angle $ATB = BCT$. The triangles ATB, ACT having two equal angles and a common angle A are similar, therefore $AB : AT :: AT : AC$, and consequently $AB \times AC = AT^2$.



COR. 3.—When the line AE passes through the centre, and when the diameter DE is equal to the tangent AT, we have

$$AT : AE :: AD : AT, \text{ or } DE : AE :: AD : AT;$$

but $DE : AE - DE :: AD : AT - AD$; that is,

$$DE : AD :: AD : AT - AD, \text{ or } AT : AD :: AD : AT - AD.$$

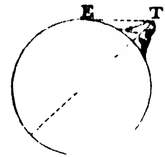
On AT lay off AG equal to AD. Then $AT : AG :: AG : GT$.

When a line AT is thus divided, that is, when the whole line is to one of the segments as that segment is to the other, it is said to be divided into *extreme and mean ratio.* Hence the mode of dividing a line AT in this manner is obviously the following:

Draw TO at right angles to AT, and make it equal to the half of AT, join AO, and from O, with the radius OT, cut AO in D, on AT lay off AG equal to AD, and AT will be divided in G, so that $AT : AG :: AG : GT$.

COR. 4.—The line AE is also divided into extreme and mean ratio in D. For $AE : DE :: DE : AD$.

EXERCISES.



1.—If the diameter of the earth be 7960 miles, and the height of a mountain on the sea coast be two miles, how far off can its top be seen at sea, the eye being supposed at the surface of the water?

2.—There are two ships whose top-masts are 90 and 120 feet above the surface of the sea, at what distance can the top-mast of the one be seen from the topmast of the other?

3.—If a mountain one mile high be seen at the distance of 89 miles, required the diameter of the globe?

4.—If when the moon is half full, a bright spot were seen at T, this spot would of course be the top of a mountain, and if the proportion between E T and the diameter of the moon were as 1 to 200, required the height of the mountain, the diameter of the moon being 2160 miles?

On these principles astronomers have determined the heights of some of the mountains in the moon, and found them higher than those on our earth. When the moon is in the form of a crescent, the rugged appearance of her surface is visible even to the naked eye; but when seen through a telescope even of ordinary powers, her dark valleys and the tops of her lofty mountains, illuminated by the rising sun, present a striking and imposing contrast.

5.—Divide the number 10 into two parts, so that the product of 10 by one of the parts shall be equal

to the square of the other part, the calculation to be founded on the property contained in Cor. 3.

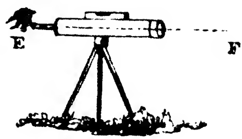
6.—It is required to produce a line, E D, (fig. page 110) to a point A, so that the whole line A E shall be to E D as E D is to the part A D produced.

7.—Required to find by calculation a number such, that if 10 be added to that number, and the sum multiplied by the number thus found, the product will be equal to the square of 10, the calculation to be founded on the properties in Cor. 3 and 4.

PRACTICAL APPLICATION OF THE PRECEDING PROPERTIES TO LEVELLING.

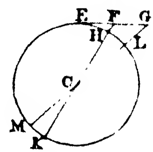
Levelling is the art of finding two or more points at the same distance from the centre of the earth. A level is an instrument by means of which a line touching the earth, or at right angles to the radius of the earth at the point of observation, may be found. The most perfect of the various kinds of levels, is what is called a *spirit level.* It consists of a telescope having a glass tube fixed parallel to its axis, and nearly filled with spirit of wine, except a small space containing air, which, being lighter than the spirit, rises to the higher end, or when the glass tube is horizontal, remains in the middle. The tin plate tube serving the purpose of a telescope for the theodolite, with a glass tube attached to it, partly filled with water, will answer for the purpose of illustration, or even for common levelling where extreme accuracy is not required.

If the glass tube be placed parallel to the axis of the telescope, and the air bubble brought to the middle of the tube, as in the annexed figure, the line E F will be horizontal.



Let E G be a horizontal line at the place E, the point C being the centre of the earth. Then since $FH \times FK = EF^2$

We have $FH = \dfrac{EF^2}{FK}$.



But for small distances, F K may be considered as the diameter of the earth. Let D be the diameter, then $FH = \dfrac{EF^2}{D}$

Ex.—When the distance E F or E H is one mile

$$\text{Then } FH = \frac{1}{7960} = 8 \text{ inches nearly.}$$

Hence, if F be one mile from E, it will be necessary to make an allowance of 8 inches below the *apparent level*, so that water may flow from E to H.

When the distance E F is two miles, then

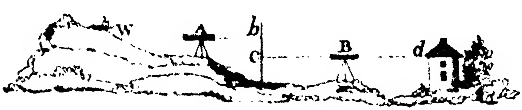
For 3 miles $FH = 3^2 \times 8 = 72$ inches
For 4 miles $FH = 4^2 \times 8 = 128$ inches.

2.—But this mode of levelling requires a very perfect and consequently an expensive instrument. It is also necessary to place the level perfectly parallel to the axis of the telescope, an operation of so delicate and laborious a nature, that very few instruments will be found perfect in this respect. The simple method employed in the following example does not require

the level to be placed exactly parallel to the axis o. the telescope, and possesses decided advantages.

Ex.—It is required to find to what height in the house at *d*, water will rise when conducted in a leaden pipe placed below the surface of the ground, from a well at w.

Place the level at A, and having brought the air bubble to the middle of the tube, direct it to a pole placed at w. Suppose the *cross-vane* on the levelling rod, to be two feet above the water in the well. Turn the level round, and direct the cross wires to another measuring rod, placed at *b*, *the distance from* A *to b being equal to* A w, then it is obvious the point *b*, and the point two feet above the well, are on the same level or at the same distance from the centre of the earth. Place the level half way between the pole *b* and the house, and repeat the same operation, then the points *c* and *d* are on the same level. If *b c* be three feet, then it is obvious that the well is one foot above *c* or *d*, and consequently water will rise to a point one foot above d.

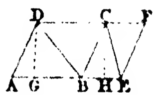


REMARK.—Such are the general principles of levelling—the practical part of which may be learned in a few hours under the direction of an experienced teacher.

SECTION V.

ON THE PROPORTIONS OF THE AREAS OF FIGURES.

PROP. I.—The areas of parallelograms having the same altitude are as their bases.



The area of the parallelogram $ABCD$ is equal to $AB \times DG$; and that of

$BEFC = BE \times CH$ or $BE \times DG$.

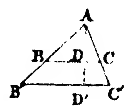
Therefore

$ABCD : BEFC :: AB \times DG : BE \times DG :: AB : BE.$

COR.—Triangles having the same altitude are to one another as their bases.

Ex.—Divide the triangle ABD into two triangles by a line drawn from D to the base, so that the parts may be to one another as 5 to 3.

PROP. II.—The areas of similar triangles are to one another as the squares of their corresponding sides.



Let ABC, $AB'C'$ be the similar triangles. Let fall the perpendicular AD on BC, which will also be perpendicular to $B'C'$.

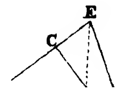
$$\begin{array}{llll} \text{Then} & AD & : AD' & :: BC : B'C'. \\ \text{Also} & BC & : B'C' & :: BC : B'C' \end{array}$$

Hence $AD \times BC : AD' \times B'C' :: BC^2 : B'C'^2$;

or by taking the halves of the first and second terms, we shall have the areas of the triangles ABC, $AB'C'$. Hence, the area of ABC : area of $AB'C' :: BC^2 : B'C'^2$. In the same manner it may be proved that the areas are as the squares of any other pair of corresponding sides.

NOTE.—This property may be shown to be merely a particular case of a more general property.

THEOREM.—If two triangles have an angle of the one equal to an angle of the other, their areas are to one another as the *product* of the sides containing that angle.



Let A B C, A D E be two triangles having the angle A common to both. Join B E, then

Area of A B C : area of A B E : : A C : A E (why ?)
Also area of A B E : area of A D E : : A B : A D (why ?)

Hence by mult. Area of A B C : area of A D E : : A C × A B : A E × A D.

PARTICULAR CASE.—When B C is parallel to D E, then
A C : A B : : A E : A D or A C × A D = A B × A E,
multiply the third term in the preceding proportion by A C × A D, and the fourth by A B × A E, and rejecting the common multipliers, we have the following proportion :

Area of A B C : area of A D E : : A C^2 : A E^2.

COR.—The areas of parallelograms having an angle in the one equal to an angle in the other, are to one another as the products of the sides containing that angle.

EXERCISES.

1.—If the area of a triangle whose base is 120 feet be 6000 square feet, what is the area of a similar triangle whose base is 250 feet?

2.—If the area of a triangle whose base is 120 feet be 6000, what is the base of a similar triangle whose area is 10,000?

3.—If the base B′ C′ of the triangle (fig. page 115), be 3240 links of the measuring chain, and the perpendicular A D′ 840 links, it is required to divide it into two equal parts by a line B C drawn parallel to the base, and to find by calculation the length of A D.

4.—Divide the same triangle into three parts, which shall be to each other as 1, 2, 3, by lines parallel to the base. Required the distances of the parallel lines from the vertex?

5.—If there be two triangles having an angle of the one equal to an angle of the other, it is required to prove that when the areas are equal, the *products* of the sides containing the equal angle, are equal. The same property belongs to equal parallelograms.

PROP. III.—The areas of similar figures having any number of sides, are to one another as the squares of any two corresponding sides.

Let ABCD, $abcd$ be the similar figures. Then since the similar triangles are as the squares of their corresponding sides, we have

NOTE.—The sign $\triangle$ is used for the word triangle.

$$ab^2 : AB^2 :: \triangle abc : \triangle ABC$$

$$\text{And } ad^2 : AD^2 :: \triangle acd : \triangle ACD.$$

But the sides ab, ad being proportional to AB, AD, their squares are also proportional,

Hence $ab^2 : AB^2 :: \triangle abc : \triangle ABC :: \triangle acd : \triangle ACD$

Conseq. $ab^2 : AB^2 :: \triangle abc + \triangle acd : \triangle ABC + \triangle ACD$

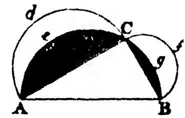
that is, as the figure $abcd$ to ABCD. The same reasoning will apply to figures having any number of sides.

COR. 1.—The areas of similar figures are also proportional to the squares of any corresponding lines drawn within them. Thus $abcd : ABCD :: ac^2 : AC^2$.

COR. 2.—Circles being similar figures, are to each other as the squares of their diameters, radii, or circumferences.

COR. 3.—If similar figures be described on the three sides of a right-angled triangle, the figure on the

hypothenuse will be equal to the sum of the figures described on the other two sides.



Let semicircles be described on the three sides, then these or any similar figures are as the squares described on the three sides, and since the square on A B is equal to the sum of those on A C and C B, it follows that the figure on A B is equal to the sum of those on A C and C B.

Cor. 4.—From the semicircle on A B, take away the two shaded segments, and there remains the triangle A C B; from the other two semicircles take away the same segments, and there remain two portions *a e* C *d*, B *f* C *g*, called *lunes*, equal to the triangle A B C.

EXERCISES.

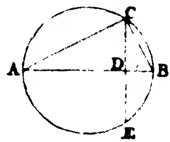
1.—The diameter of a circular fish-pond is 12 feet, required the diameter of another whose area shall be nine times that of the other.

2.—Required to divide a circle whose diameter is 100 feet into two equal parts by the circumference of another circle described from the same centre. With what radius must the second circle be described?

3.—There is a circular fish-pond whose diameter is 40 feet, it is required to find what must be the breadth of a circular gravel-walk round it, so that the area of the walk may be one-fourth of that of the pond?

Prop. III.—If a perpendicular be let fall from the right angle of a right-angled triangle on the hypothenuse, it will divide the triangle into two triangles, which are similar to the first and to each other.

Let ACB be the triangle, and let a circle be described about it, and the perpendicular from C, the right angle, produced to E.



m ——

n ——

Then $\angle$ CAB $=$ BCE (why?) and
$\angle$ ACD $= \angle$ DBC (why?) therefore the triangles ACD, CBD and ACB are similar.

COR. 1.—Since AD : AC :: AC : AB

$$AD \times AB = AC^2 \quad . \quad . \quad . \quad (1)$$

$$BD \times AB = BC^2$$

COR. 2.—Since $AD \times AB : BD \times AB :: AC^2 : BC^2$
by dividing the first and second terms by AB we have

$$AD : BD :: AC^2 : BC^2 \quad . \quad . \quad . \quad (2)$$

COR. 2.—Since $AD \times AB : BD \times AB : AB \times AB :: AC^2 : BC^2 : AB^2$.
Dividing by AB we have $AD : BD : AB :: AC^2 : BC^2 : AB^2$. That is, if a perpendicular be let fall from the vertex of the right angle on the hypothenuse, the squares described on the two sides and on the hypothenuse are to each other as the adjacent segments of the hypothenuse and the hypothenuse itself. This may be deduced in a different manner from the Second Demonstration of the 47th of the First Book of Euclid.

COR. 4.—From the property contained in Cor. 1, may be deduced an elegant demonstration of the celebrated 47th of the First Book of Euclid's Elements.

Since $\quad AB \times AD = AC^2$

And $\quad AB \times BD = BC^2$

By addition $\quad AB \times AD + AB \times BD = AC^2 + BC^2$

Or $\quad AB \times \overline{AD + BD} = AC^2 + BC^2$

That is $\quad AB \times AB$ or $AB^2 = AC^2 + BC^2$.

The pupil will be pleased to see this important property presented to him in so many different points of view.

EXERCISES.

1.—The diameter A B of a circle is 100, and the length of the chord A C 60, required the length of the segment A D by calculation.

2.—Required by calculation the sides of two squares, whose areas shall be to one another as 9 to 16, and the sum of the areas 400 square inches?

3.—Required to find by construction the sides of two squares, whose areas shall be to each other as two lines *m*, *n*, and having the sum of the squares equal to the square described on a given line A B (last fig.)

4.—Required to find by calculation the sides of two squares, whose areas shall be in the ratio of 9 to 25, that the difference between their areas shall be 256 square miles.

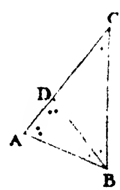
5.—Required to find by construction the sides of two squares, whose areas shall be to one another as the two lines *m*, *n*, and the difference between them equal to a square described on a given line A C (last figure.

SECTION VI.

ON CERTAIN PROPERTIES BELONGING TO THE REGULAR PENTAGON AND DECAGON, WHICH COULD NOT HAVE BEEN EASILY INVESTIGATED IN PART I.

Prop. I.—Prob.—To determine the conditions on which the construction of a regular *decagon* depends.

ANALYSIS.



Let A C B be an isosceles triangle, having the angle C the $\frac{1}{10}$ part of 360 degrees, or $\frac{1}{10}$ part of four right angles, or $\frac{4}{10}$, or $\frac{2}{5}$ of one right angle; then it is obvious, that if a circle be described with the radius C A, the chord A B will go exactly 10 times round the circumference, and form a regular decagon.

Since the angle C is $\frac{2}{5}$ of a right angle, or $\frac{1}{5}$ of two right angles, the remaining angles C A B, C B A, of the triangle A C B, must be $\frac{4}{5}$ of two right angles. But these angles being equal, each of them is $\frac{2}{5}$ of two right angles, and consequently double of the angle at C. Bisect $\angle$ A B C by B D, then the triangle B D C is isosceles, and consequently its exterior angle, A D B, is double of the angle C, or equal to the angle A. Hence A B D is an isosceles triangle, and similar to A C B.

Hence A C : A B : : A B : A D; but A B = B D = C D, therefore A C : C D : : C D : A D. Hence, if C A, the radius of a circle, be divided into extreme and mean ratio at D, then the greatest segment, C D, will be the side of the inscribed

SYNTHESIS.

Case I.—Given the radius of a circle; it is required to find the side of the inscribed decagon.

Divide the radius into extreme and mean ratio, and the greater segment will be the side of the decagon required.

Ex.—The pupil is required to do this by actual construction.

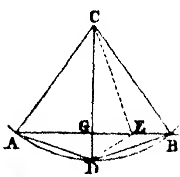
CASE II.—Given the side of a regular decagon; it is required to construct it, or determine the radius of the circumscribing circle.

Let A B be the line on which it is required to construct a regular decagon. Produce A B so that the rectangle contained by the whole line produced and the part produced, shall be equal to the square of A B; then the whole line produced will be the radius of the circumscribing circle.

Ex.—The pupil is required to do this by actual construction.

COR.—If the alternate angular points of the decagon be joined, we shall have the corresponding *pentagon*.

PROP. II.—The square described on the side of a regular pentagon inscribed in a circle is equal to the sum of the squares of the radius and the side of the inscribed decagon.

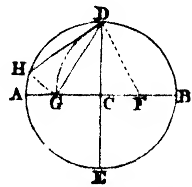


Let A D, D B be sides of the inscribed decagon; join A B, which will be the side of the inscribed pentagon. Bisect ∠ B C D by C E, then the triangles C D E, C B E, are equal; (why?) therefore D E = E B, and consequently the triangles B E D and A D B are similar. Hence A B : B D :: B D : B E, or $BD^2 = AB \times BE$. (1).

Again; since the radius bisects the angle of a regular figure, the angle C A E is half one of the angles of the inscribed pentagon, and is consequently $\frac{3}{5}$ of a right angle; (why?) But the angle A C E is also $\frac{3}{5}$ of a right angle, since A C D is $\frac{2}{5}$, and D C E $\frac{1}{5}$ of a right angle. Hence A E C is an isosceles triangle, and similar to A C B. Hence A B : A C :: A C : A E, or $AC^2 = AB \times AE$. (2).

Adding equation (1) to (2) we have $BD^2 + AC^2 = AB \times BE$

Cor.—Hence the following practical method of inscribing a pentagon in a given circle.



Bisect the radius CB in F, join FD, and make $FG = FD$; join GD, which will be the side of the pentagon required.

For by the slightest reference to the mode of dividing a line into extreme and mean ratio, the pupil will see that AC is so divided in the point G, CG being the longer segment. From this the pupil is required to show that GD is the side of the pentagon required.

SECTION VII.

ON THE AREAS OF REGULAR FIGURES AND CIRCLES.

If lines be drawn from the centre of a circle circumscribing any regular figure to each of the angular points, the figure will be divided into as many equal triangles as the figure has sides. By determining the area of one of the triangles the area of the regular figure will thus be obtained.

Prop. I.—Prob.—To find the area of an equilateral triangle whose side is 1.

If a perpendicular be let fall from the vertex on the base, it will bisect the base; hence the perpendicular $= \sqrt{1^2 - \cdot 5^2} = \sqrt{.75} = .866025$;

consequently the area $= \dfrac{1 \times .866025}{2} = .433012$.

Prop. II.—Prob.—To find the area of a regular hexagon whose side is 1.

Since the side of the hexagon is equal to the radius of the circumscribing circle, and since a perpendicular from the centre of the circle on one of the sides bisects that side, we have this perpendicular the same as for the equilateral triangle, or .866025, and the area of the triangle $=.433012$; hence the area of the hexagon $=.433012 \times 6 = 2.59807$.

Prop. III.—Prob.—To find the area of a regular octagon whose side is 1.

By examining the mode of constructing a regular octagon (fig. page 59) we have $\text{A C} = \text{C D}$, and $\text{A D} = \sqrt{\text{A C}^2 + \text{C D}^2} = \sqrt{.5^2 + .5^2} = \sqrt{.50} = .7071$;

And $\text{C E} = \text{E D} + \text{D C} = .7071 + .5 = 1.2071$;

Hence area of triangle $\text{A E B} = \dfrac{1 \times 1.2071}{2} = .6035$,

And area of octagon $= .6035 \times 8 = 4.828$, &c.

Note.—If the areas of all regular figures, whose side is 1, were determined, the areas of similar figures, whose sides were expressed by any numbers, could easily be found.

EXERCISES.

1.—What is the area of an equilateral triangle whose side is 5 feet?

2.—What is the area of a regular hexagon, whose side is 10 feet?

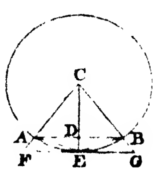
3.—What is the area of a regular octagon whose side is $3\frac{1}{3}$ feet?

General Exercise.—The side of a regular decagon inscribed in a circle is 1; required the length of the radius of the circle, the length of the perpendicular let fall from the centre on one of the sides, and the area of the decagon. Required also the length of the

side of the inscribed pentagon, the perpendicular let fall from the centre on the side of the pentagon, the area of the pentagon, and the area of a pentagon whose side is 1.

NOTE.—This exercise, being rather difficult for beginners, may be passed over at first reading.

PROP. IV.—PROB.—To find the length of the circumference whose diameter is 1.



Let A B be the side of the inscribed hexagon, which being equal to the radius, is $\frac{1}{2}$, or .5. Let fall the perpendicular C D and produce it to E; join A E, E B. Hence the perpendicular C D $= \sqrt{.5^2 - .25^2}$, consequently D E $=$ C E $-$ C D, &c.; and A E $= \sqrt{\text{A D}^2 + \text{D E}^2}$.

In like manner, by bisecting A E by a perpendicular from C, and going through the same form of calculation, the length of one of the sides of an inscribed regular figure having 24 sides may be determined. In like manner, the lengths of the sides of regular inscribed figures having 48, 96, 192, 384, &c. may be determined.

Having determined the perimeter of an inscribed regular polygon, the perimeter of the circumscribing one having the same number of sides, may easily be determined. Draw a tangent at E, meeting C A, C B, produced in F and G, then F G is the side of the circumscribing polygon having the same number of sides.

Then, since C D : C E :: A B : F G, we have C D to C E as A B multiplied by the number of sides to F G multiplied by the same number, that is C D : C E :: perimeter of the inscribed polygon : perimeter of the circumscribing polygon.

But taking two polygons with a great number of sides, the perimeters of the inscribed and circumscribing figures will be found the same to the fifth or sixth decimal place. Hence the circumference of the circle, which is greater than the

perimeter of the one and less than that of the other, will also be correct to the same decimal figure.

In this way the circumference of a circle whose diameter is 1, will be found to be 3.14159, &c.

Hence, since circles are similar figures, the circumference will be found by multiplying the diameter by 3.14159.

Since 22 divided by 7 gives 3.142, the circumference of a circle may be found for common purposes by multiplying the diameter by 22 and dividing by 7.

EXERCISES.

1.—The diameter of a circle is 12; required its circumference.

2.—The circumference of a circle is 100 feet; required its diameter.

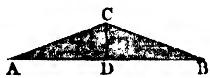
PROP. V.—PROB.—To find the area of a circle.

If the circumference be conceived to be divided into an indefinitely great number of equal parts, and lines drawn from the centre to each division, the indefinitely small portion of the arc may be viewed as a straight line, and the whole circle may be considered as divided into the same number of triangles. Now the area of one of the triangles will be found by multiplying the length of the arc or base by half the perpendicular or radius, and consequently the sum of all the triangles, or the area of the circle, will be found by multiplying the sum of all the bases, or the whole circumference, by half the radius.

COR. 1.—The area of a circle is therefore equal to that of a triangle, having its base equal to the circumference, and its perpendicular equal to the radius, or to a rectangle having one side equal to half the circumference, and the other equal to the radius.

POPULAR ILLUSTRATION OF THE FIRST PROPERTY.—Take a circle of paper, and divide the circumference into a number of equal parts, suppose 24; divide into two semicircles, and from the centre

draw the several radii. By means of a sharp knife cut the whole into *sectors* or portions, bounded by two radii and the intercepted arc. Half the number of sectors being placed on a board with the arcs touching a straight line, will appear as in the annexed figure, and the other half placed so as to fill the vacant spaces, will form a figure resembling a rectangle, having one of its sides, A B, equal to half the circumference, and the other side, B C, equal to the radius. The greater the number of parts into which the circle is divided, the more nearly will the figure approach to a real rectangle.*



POPULAR ILLUSTRATION OF THE SECOND PROPERTY.—Take a ribbon of paper, about half an inch broad and several feet long, roll it about a fine wire, in the same way as ribbons are rolled on cylinders, the end of it will then form a circle. By means of a sharp knife cut through from the circumference to the axis or centre, and open out the several portions in a straight line, and the circular end will be changed into the annexed triangular figure, in which A B is the whole circumference, and the perpendicular C D the radius.

COR. 2.—Hence the area of a circle whose diameter is 1 =

COR. 3.—Since circles are similar figures, let d be the diameter of any circle, then

$1^2 : d^2 :: .7854$: area of the circle whose diameter is d.
Hence, to find the area of a circle, multiply the square of the diameter by .7854.

COR. 4.—The area of a sector containing any number of degrees will be obviously found as follows:

360 is to the number of degrees in the arc of the sector as the area of the circle to that of the sector.

EX.—If the radius of the circle (fig. page 125) contain 20

* Asiatic Researches.

feet, and the number of degrees in the arc A E B be 60, required the area of the section C A E B.

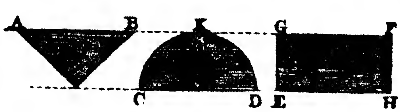
General Exercise.—The pupil is required to find the circumference of a circle whose diameter is 1, by beginning with the inscribed square, and finding in succession, the side of the inscribed octagon, &c. till the perimeter of the inscribed polygon be found to be 3.1415, &c. and that of the circumscribing one having the same number of sides, 3.1416, &c.

SECTION VIII.

ON THE CYLINDER, CONE, SPHERE, AND THE FIVE REGULAR SOLIDS.

1.—It was shown in the first part of the volume that prisms, standing on equal bases and having the same altitude, are equal, and that all pyramids having equal bases and altitudes, whatever may be the form of the base, are also equal to one another. When the base on which the prism stands is a circle, the solid is called a *cylinder*, and when the base of the pyramid is a circle the solid is called a *cone*. Hence the solidity of a cylinder will be found by multiplying the *area* of the base by the perpendicular height, and that of the cone by taking $\frac{1}{3}$ of the cylinder having the same base and altitude.

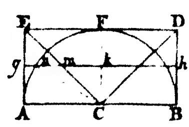
2.—General Principle.—If there be three solids, suppose a *cone*, a *hemisphere*, and a hav-



ing the same altitude, or standing between the same planes, and having the diameters of the bases, C D, E H, and the diameter A B of the base of the inverted cone, all equal; and if these solids be conceived to be made up of the same number of indefinitely thin planes, or laminæ, and if it be proved that the sum of the areas of the planes which compose the cone and hemisphere is at the same altitude always equal to the area of the corresponding plane belonging to the cylinder, it follows that the cylinder is equal to the sum of the cone and hemisphere; for the laminæ having the same thickness, the sum of the two laminæ belonging to the cone and hemisphere will be equal in solidity to the corresponding lamina belonging to the cylinder, and as the number is the same, the solidity of all the laminæ composing the cone and hemisphere will be equal to those which make up the cylinder.

APPLICATION OF THIS PRINCIPLE.

THEOREM.—If a cone, a hemisphere, and a cylinder stand on equal bases, and have the same altitudes, they are to each other as 1, 2, 3.

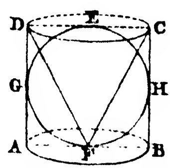


Let A B D E be a section of a cylinder, A F B that of a hemisphere, and E C D that of a cone, formed by a plane dividing each of them into two equal parts. Let the three solids be conceived to be cut by another plane, perpendicular to the former in the line $g\,h$. Then in the right-angled triangle $c\,k\,n$ we have $c\,n^2 = c\,k^2 + n\,k^2$, but since C F E is an isosceles triangle, and consequently $c\,k\,m$, $c\,k = m\,k$, and since $c\,n = A\,C = k\,g$, we have $k\,g^2 = k\,m^2 + k\,n^2$; but circles being to each other as the squares of their radii, we have the area of the circle, whose radius is $k\,g$, equal to the sum of the areas of the circles whose radii are $k\,m$ and $k\,n$. But $k\,g$ is the radius of the circular

lamina belonging to the cylinder at the height c k, and $k m$, $k n$, are the radii of the circular laminæ belonging to the inverted cone and the hemisphere at the same height. Hence the whole cylinder is equal to the cone and hemisphere taken together. But the cone is $\frac{1}{3}$ of the cylinder, therefore the hemisphere is $\frac{2}{3}$; consequently the three solids are to one another as 1, 2, 3.

Cor.—If we take the complete sphere, we shall double the heights of the cone and cylinder, whilst their bases remain the same, and consequently double their solidities; hence a sphere is $\frac{2}{3}$ of its circumscribing cylinder.

Note.—We are indebted to the genius of Archimedes, the greatest mathematician of antiquity, for the discovery of these beautiful relations.



Obs.—This beautiful relation might easily be shown to those who may find any difficulty in following the preceding process of reasoning, by taking a cylinder, A B C D, made of tinplate, and having the same diameter and height as a globe of wood, E G F H, and also a cone of wood, C F D, having the same base and height. If the cylinder be filled with water, and the cone immersed, $\frac{1}{3}$ of the water will flow over; if the cone be removed, and the same experiment repeated with the globe, $\frac{2}{3}$ of the whole water will now flow over. Hence, by experiment, the solids are to each othe as 1, 2, 3.

Hence the following rule for finding the solidity of a sphere.

Rule.—Multiply the cube of the diameter by .5236.

For the solidity of a cylinder whose diameter we may call d, and whose height is the same, is $= d^2 \times .7854 \times d$, or $d^3 \times .7854$. If we take $\frac{2}{3}$ of this, we shall have d^3 multiplied by $\frac{2}{3}$ of .7854, which is .5236.

Cor.—Hence spheres are to one another as the *cubes* of their diameters. For the solidity of one sphere is to that of

as the cube of the diameter of the first multiplied by .5236, is to the cube of the second multiplied by .5236. Hence, taking away the common multiplier, the solidities are as the cubes of their diameters.

Ex.—If a globe of silver whose diameter is 2 inches be worth £5, what is the diameter of that globe whose value is £1350?

PROP. II.—To determine a rule for finding the surface of a sphere.

If the whole of the surface of the sphere be conceived to be divided into an immense number of very small squares, and planes be conceived to pass through the sides of the squares and the centre of the sphere, the solid will be divided into as many pyramids, having their vertices in the centre, as there are squares. Now the solidity of one of these pyramids will be found by multiplying the area of the square by $\frac{1}{3}$ of the radius or height, and the sum of all the pyramids, or the solidity of the sphere, will be equal to the whole surface of the sphere multiplied by $\frac{1}{3}$ of the radius. But the solidity is also expressed by $\frac{2}{3}$ of the circumscribing cylinder. Hence if we call s the surface, and d the diameter, we have

$$s \times \frac{\text{rad.}}{3}, \text{or } s \times \frac{d}{6} = \tfrac{2}{3} \text{ of } (d^2 \times .7854 \times d).$$

Dividing both sides by d, we have $s \times \frac{1}{6} = \frac{2}{3}$ of $d^2 \times .7854$. Hence since the sixth part of the surface is equal to $\frac{2}{3}$ of $d^2 \times .7854$, the whole surface must be six times this quantity, or $4 \times d^2 \times .7854$; but $d^2 \times .7854$ is the area of one of the great circles of the sphere; *hence the surface of a sphere is equal to 4 times the area of one of its great circles.*

EXERCISES.

1.—If the diameter of the earth be 8000 miles, how many square miles are contained on its surface?

2.—If the diameter of the earth be 8000 miles, and that of the sun 890,000, how many times is the sun greater than the earth.

III.—On the *five* regular solids.

DEF. 1.—A *regular solid* is one bounded by equal and similar regular plane surfaces.

2.—A *solid angle* is contained by three or more plane angles meeting in a point. It is obviously necessary that the sum of the degrees in all the plane angles be less than *four* right angles, for if equal the plane angles would all lie in the same plane.

3.—The regular solid having the least number of sides, is that bounded by four equilateral triangles, called a *tetrahedron*. This being a pyramid, its solidity is found by a rule given in Part I.

4.—The next regular figure is that bounded by six squares, called a *hexahedron* or *cube*.

5.—An *octahedron* is bounded by eight equilateral triangles. This solid being made up of two square pyramids having their base joined, may have its solidity calculated by the rule for a pyramid.

6.—The *dodecahedron* is bounded by *twelve* regular pentagons.

7.—An *icosahedron* is bounded by *twenty* equilateral triangles. Each solid angle is made up of five plane angles.

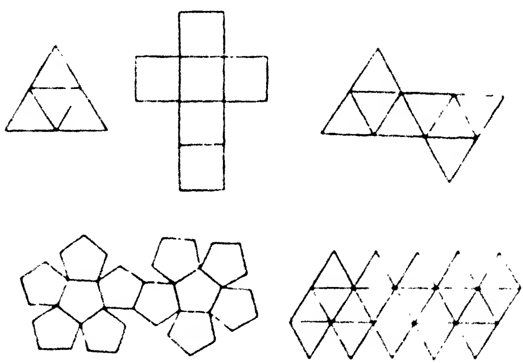
8.—These may easily be shown to be the only regular solids which can be found.

If it were attempted to make a regular solid which should be bounded by regular hexagons, we should find that *three* of the plane angles being joined, would make up 360 degrees, or lie on the same plane. Hence a regular solid cannot be bounded by hexagons.

If we attempted to make a regular solid bounded by more than twenty equilateral triangles, we must have *six* of the plane angles meeting in a point, and forming 360 degrees, or all lying on the same plane.

Ex.—The pupil is required to show that no other solids can be formed bounded by *squares* or *pentagons*.

Note.—As the pupil ought to be well acquainted with the forms of these solids, he may easily make them for himself, by cutting a piece of pasteboard, or drawing paper into the following figures.



If the lines be cut with a sharp knife nearly through the paper, and the different plane surfaces folded at those lines and turned up, the edges will all meet and form the regular figures in order. The edges may then be secured by paste or glue.

Note.—The full investigation of these figures is rather too difficult for a work of so elementary a nature as the present. When the pupil is a little farther advanced, he will find the subject fully inves-

tigated in Hutton's larger treatise on the Mensuration of Solids.

Obs.—It is sufficiently obvious that regular figures of the same kind are perfectly *like*, or similar to each other, whatever may be their size, and like spheres may be proved to be to one another as the *cubes* of their corresponding lineal dimensions. This property belongs to all similar solids, but the investigation of this and other properties would extend the volume beyond the limits prescribed.

Obs.—It was not my intention to have entered at all on the Geometry of Solids, but after more mature deliberation, it was thought advisable to devote two short sections to this subject, that the pupil might, at an early period of his studies, obtain clear ideas of the most essential properties of those solids which are most frequently employed. What is here offered, is therefore to be viewed only as a short introduction to the *geometry of space*.

GENERAL OBSERVATIONS.

In the Second Part of this Volume, we have endeavoured to simplify the subject, and in avoiding the introduction of whatever is not essential in establishing the most important properties, have necessarily passed over what are called *incommensurable quantities*, or those which cannot be represented by finite numbers. Suppose, for example, we wanted to compare the length of the side of a square with its diagonal. If the side of the square be 1, the diagonal will be expressed by the square root of 2, which cannot be expressed in finite numbers, but will turn out 1 with an interminate decimal fraction. We say then, the side of a square is incommensurable with its diagonal.

The perpendicular of an equilateral triangle, whose side is 1 equal to the square root of $1\frac{1}{4}$, which cannot be extracted in finite numbers. The perpendicular of an equilateral triangle whose side is 1, is therefore incommensurable with its side. In like manner it may be shown, that if the side be expressed by any finite number, the perpendicular cannot be expressed in finite decimals.

Hence as the ratio of the diameter of a circle to its circumference, proceeds on finding the perpendicular of an equilateral triangle when the side is given, or of finding a side of a square when the diagonal is given, it follows that the circumference of a circle, whose diameter is a finite number, cannot be determined in

finite numbers. As the area of a circle necessarily depends on the ratio of the diameter to the circumference, and as this cannot be expressed in finite numbers, neither can the area be expressed in finite parts of the circumscribing figure.

When the diameter is 1, the circumference is found to be 3.14159265358979323846, &c.*

This popular view of the nature of incommensurate quantities will be sufficient for the general scholar. The mathematical student, when he advances farther, will view the subject in a different light, and examine it more in detail than is consistent with the plan of this volume.

PROMISCUOUS EXERCISES.

As nothing tends more to make the knowledge which a pupil has acquired *really his own*, than the application of that knowledge to the investigation of theorems and the solution of problems, a few simple exercises of a miscellaneous nature will be found useful.

1.—In a given circle it is required to inscribe a triangle, whose angles shall contain 30, 70, and 80 degrees respectively, without drawing a tangent to the circle. The property to be employed is, that angles in the same segment are equal to one another.

2.—In a given equilateral triangle, it is required to inscribe a square. The investigation to proceed on the principles of Geometrical Analysis.

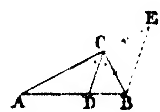
* This number has been calculated to above one hundred decimal places.

3.—If earth dug out in forming a circular pond, whose diameter is 120 feet, and depth 6 feet, be spread over a field to the depth of $2\frac{1}{2}$ inches, it is required to know what space it will cover on the supposition that the surface is uniformly covered.

4.—There is a rectangular garden 60 feet long by 40 broad, it is required to find the breadth of a walk to be made round it, so that the walk shall occupy $\frac{1}{16}$ of the whole rectangle.

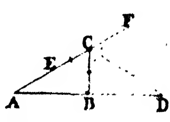
If the pupil proceed by supposing the rectangle converted into a square, whose side is 50, or perimeter 200, the same as that of the rectangle, he will find no difficulty in determining the breadth of the walk.

5.—If one angle of a triangle ACB be bisected by a straight line CD cutting the base, it is required to prove that



$$AC : CB :: AD : DB.$$

6.—If one side of a triangle be produced, and if the exterior angle be bisected by a line cutting the base produced in D, it is required to prove that



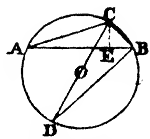
$$AC : CB :: AD : BD.$$

7.—Cut off from an extremity of a given line $\frac{3}{7}$ of the whole line.

8.—Given the three sides of a triangle to determine the diameter of the circumscribing circle.

Let fall the perpendicular C E on A B, draw C D diameter of the circle, and join D B. Then the triangle C A E is similar to C D B, (why?)

Therefore A C : C E :: C D : C B.



Ex.—Let A B = 100, A C = 90, and C B=50, required the diameter C D of the circumscribing circle.

9.—It is required to construct a triangle, the sum of whose sides shall be represented by 500, and the three angles 30, 40, and 110 degrees respectively. Find out the construction by analysis, and give the lengths of sides as nearly as possible from accurate construction.

10.—If the sides of any four-sided figure be bisected, and the points of bisection joined, it is required to demonstrate that the figure thus formed is a parallelogram.

11.—If the diameter of the larger end of a tapering tree be 20 inches, that of the smaller end 10, and the length 12 feet, it is required to find the length of the portion cut off, on the supposition that the tree tapered to a point. Required also the number of cubic feet in both portions of the tree.

12.—Divide a right angle into *three* equal parts without using the protractor or line of chords.

13.—If the diameter of the sun be 890,000 miles, it is required to find by accurate construction, at what distance from its centre a person must be placed so that the sun may be seen subtending an angle of 25 degrees? The investigation to be conducted by geometrical analysis.

14.—If a cubic foot of brass weigh 8544 ounces avoirdupois weight, required the diameter of a brass wire 100 feet long which weighs one ounce?

15.—To find the side H K of a rectilineal figure, which shall be similar to A B C D and equal to another E F G.

ANALYSIS.

Suppose it done, and that ab is the side required, which corresponds to A B.

Then area of A B C D : area of similar fig. on H K :: A B^2 : ab^2.

Convert the given figures into squares, and let m be the side of the square equal to the first, and n that of the second.

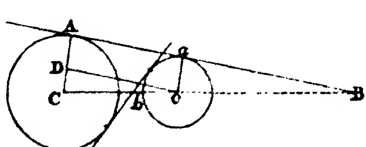
$$\text{Then} \quad m^2 : n^2 :: AB^2 : ab^2$$
$$\text{Or} \quad m : n :: AB : ab$$

But m, n and A B are given, therefore ab is also given.

Synthesis, to be given by the pupil, and the mode of construction fully described.

16.— Given two circles in position and magnitude, it is required to draw a line which shall touch the circumferences of both circles.

Case. I.—When the tangent A a is on the same side of the line joining the centre.



ANALYSIS.

Join C A, $c\,a$, and draw C D parallel to A a. Then C D = C A — $c\,a$, and is therefore given. C c is also given, and the angle C D c is a right angle, (why ?) Hence the triangle C D c is given, and consequently C D is given in position, and its intersection A with the circumference is also given, and consequently the position $c\,a$, and its intersection a with the circumference is also given. Hence the tangent A a is given.

SYNTHESIS.

The pupil is required to give the construction and demonstration synthetically. He is also required to give the analysis and synthesis of the case where the line touches the circles on opposite sides of the centre. He is also to investigate both cases by employing the proportional properties of parallel lines, and determine the points B, b, where the tangents cut C c or its production.

17.—If the diameter of the sun be 890,000, that of the earth 8000, and the distance between them 95,000,000 miles; required to find the distance of the vertex of the *conical* shadow of the earth from its centre.

18.—If the diameter of the moon be 2160 miles, and that of the sun 890,000 miles, and the distance between them 95 millions of miles, it is required to find by calculation at what distance from the moon an observer must be placed so that that the sun may appear totally eclipsed, and no more, by the interposition of the moon?

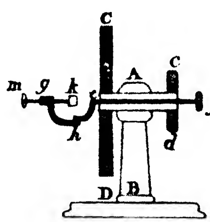
19.—There is a vessel in the form of a portion or *frustum* of a cone, the diameter of the bottom is 140 inches, of the top 180, and the depth along the slanting side 120 inches, how many gallons does it hold, a gallon containing 231 cubic inches?

20.—Divide a circular piece of ground whose diameter is 270 feet, into three portions, which shall be to one another as 1, 5, 3, by concentric circles, the outer ring being for a walk, the second for a flower-garden, and the inner circle for a pond. Required the lengths of the radii with which the two circles must be described?

APPENDIX.

As nothing tends more to inspire youth with a love for knowledge than being made acquainted with the various uses to which it may be applied, I shall now show how the simple theodolite, formerly described, may, by a few slight additions, be converted into two of the most elegant instruments for measuring angles which we possess. One of them leading the pupil on to the study of *Mineralogy*, the other to *Navigation* and *Astronomy*.

I.—Description and application of Wollaston's Goniometer for measuring the angles of crystals.

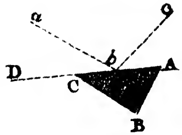


Let A B be an upright pillar standing on a solid base or sole of wood, for the purpose of supporting the instrument. Let *c d* represent the head of an axis of wood which passes through a hole in the top of the pillar, the other end of which is pushed into the centre of the circle, of which C D is a section, and which was before described as a theodolite. A strong brass wire or rod *f e* passes through the axis, and has fixed to it a piece of brass *e h g*, having a joint at *h*. Another brass wire *m k* passes through a tube *g*. A thin piece of brass *k* goes into a slit in the end of the wire on which a small crystal, for example, a piece of quartz, may be fixed by means of

bees-wax or soft cement. A cross piece of wood fixed to the pillar, and having two verniers on its ends for the purpose of reading off the number of degrees aud minutes which the circle turns round, completes the instrument. Instead of having the index fixed, and the circle moveable, as in the common form of the goniometer, the circle may be fixed to the pillar, and the index, carrying the verniers, moveable, on the side of the instrument where the crystal is fixed.

USE.—The circle of the instrument being supposed to be placed in a vertical plane, and the crystal fixed at *k*, so that the line where the two planes of the crystal meet may be nearly horizontal, and in the continuation of the axis of the instrument or parallel to it, we place it accurately so in the following manner. The instrument being placed opposite a house, we fix on any two remarkable lines which are both horizontal, for example, the top and bottom of two windows, or we may use the cross bars in one of the windows. The eye being then placed near the crystal, we view by reflection the horizontal line of the highest window; we then gradually turn the crystal by means of the nut *f*, till we bring this horizontal line, seen by reflection, opposite another horizontal line seen directly. If the two lines coincide, the plane is correctly placed; if not, the position of the crystal must be changed by some of the movements at *h*, *g* or *k*, till this be accomplished. The same operation must be gone through with the adjacent plane, till both planes fulfill this condition, and their intersection will then be horizontal.

This adjustment being made, and the circle turned till the index stand at zero on one of the verniers, we again place the eye near the crystal and turn the nut *f* till we see the reflected image of the upper horizontal line coinciding with the lower on one of the faces of the crystal. We then turn the nut *c d*, which carries the circle and crystal along with it, till we see the same coincidence on the adjacent plane of the crystal. We then observe the number of degrees and minutes through which the circle has passed, the *supplement* of which will be the inclination of these planes to one another.



For let A C B be a section of the crystal, and A C B be the angle formed by two of its planes. Let $a\,b$ be the ray of light from the higher line, and which after reflection will enter the eye at c. Let the crystal be now turned round till the side C B come into the position of C A, then it is obvious that the crystal must have gone through an angle equal to D C B, which is the supplement of A C B, the angle formed by the planes of the crystal.

Ex.—If the circle, and consequently the crystal, has passed through 110° 15′ before the reflected object and that seen directly coincide on both faces of a crystal, required the angle formed by its planes?

Obs.—If the teacher procure a small triangular prism of glass, he may allow each pupil to determine the angles formed by the three planes, the sum of which, if the observations have been correct, will be nearly 180°. In case the instrument should not be correctly centred, the number of degrees and minutes through which the circle passes, as given by each of the verniers, should be added together and half their sum taken as a good approximation to the truth. The same remark applies to the next instrument.

II.—Description and application of the *reflecting circle* for measuring angles.

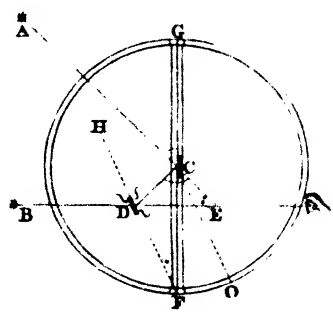
The circle which we have employed as a theodolite and goniometer, may, with a small addition, be made to supply the place of *Hadley's quadrant* or *sextant*.

The instrument is founded on the following properties:—

1.—When light falls on a plane, polished surface, it is reflected at an angle equal to the angle of incidence.

2.—If a triangle have one of its exterior angles and one of its interior opposite, respectively double of those in another triangle, its remaining interior opposite angle will also be double of the corresponding one in the other triangle. For since the exterior angle of a triangle is equal to the *two* interior opposite angles, it follows that if we double the exterior angle of a triangle, and also double *one* of its interior opposite angles, we also double the remaining one.

Description.—Let G F be a flat piece of wood, having two verniers, one at G and the other at F. Let C be the centre of the circle, directly above which is placed a piece of plane mirror, represented by the black line. This mirror is placed at right angles to the plane of the instrument by means of three adjusting screws; another piece of plane mirror, D, having the lower half silvered and the upper half transparent, is fixed at D, and adjusted so as to be parallel to the other when the *zero* of the vernier is brought to O on the circumference or *limb* of the instrument.



Let A C be a ray of light from an object A. This ray will be reflected in the direction C D, and then in the direction D E, entering the eye at E, which receives the direct light from the object B through the transparent part of the glass. Hence the object A, seen by reflection, will coincide with the object B, seen directly. Let the various lines be drawn as in the figure. Then in the triangle C D E its exterior angle, C D B, is double of the exterior angle C D H in the triangle C D F; (why?) Also, in the triangle C D E one of its interior opposite angles, D C E, is double of one of the in-

terior opposite angles, D C F, in the triangle C D F (why?) Therefore the remaining interior opposite angle, C E D, in the first, is double of the remaining interior opposite angle, C F D, in the second. But the angle C F D is the inclination of the two mirrors, and this angle being equal to O C F (why?) it follows that the number of degrees which the index has passed over from zero will be half the number of degrees in the angle A E B, formed by lines drawn from the eye to the distant objects A, B. Hence, in using this instrument we must *double* the arc over which the index passes.

OBS.—Before using this instrument the *index* glass C must be placed at right angles to the plane of the instrument, which is done by observing whether the divided circle on the instrument, seen by reflection in C, coincides with the same circle seen directly. If they do not coincide, the screws must be slightly altered, till by trial coincidence is found to take place. The *horizon* glass D being also placed vertical, the index is brought to zero, and a distant object, B, is viewed by reflection, and also directly; if the reflected image coincide with the object seen directly, the glasses are parallel, if not, D must be slightly altered till this condition be obtained. It is obvious the eye may be placed anywhere in the line D E or its production.

To measure the angle formed by two distant objects A, B. Place the index at *zero*, and look at the object A, which will be seen directly through the transparent part of the glass D, and also by reflection in the silvered part. Turn the instrument and the index in opposite directions, so as to keep the object A seen by reflection in D constantly in view, until that reflection coincide with B seen directly through the transparent part of the glass; then *twice* the arc over which the index has passed, will give the degrees in the angle A E B.

By using an artificial horizon of mercury, or even soft treacle, in a flat saucer, the image of the sun seen by reflection in the instrument, may be brought to coincide with his image seen by reflection in the artificial horizon, and his altitude at any hour ascertained.

Concluding Observation.—The pupil, ardent in the pursuit of elegant and useful knowledge, having thus clearly seen a few of the beautiful and important applications of Elementary Geometry, will now feel impatient to acquire a knowledge of the other branches of the mathematics, from the application of which the most splendid results have been obtained.

FINIS.

N. B. The author being firmly convinced of the vast importance of the simple INSTRUMENTS described in this volume, has made arrangements that teachers or families may be supplied with them on application to the Publisher.

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